# QUASI-CONFORMAL MAPPINGS AND PERIODIC SPECTRAL PROBLEMS IN DIMENSION TWO

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ABSTRACT. We study spectral properties of second order elliptic operators with periodic coefficients in dimension two. These operators act in periodic simply-connected waveguides, with either Dirichlet, or Neumann, or the third boundary condition. The main result is the absolute continuity of the spectra of such operators. The corner stone of the proof is an isothermal change of variables, reducing the metric to a flat one and the waveguide to a straight strip. The main technical tool is the quasi-conformal variant of the Riemann mapping theorem.

## 1. INTRODUCTION

According to a common belief, second order elliptic differential operators with periodic coefficients should not have degenerate bands in their spectra, or, in other words, their spectra should be purely absolutely continuous (see [8], [19], [32]). The first rigorous proof of this fact was given by L. Thomas in [33] for the Schrödinger operator $-\Delta + V$ with a periodic real-valued potential $V$. Further developments in this area were driven by the attempts to consider operators with ever increasing "strength" of the periodic perturbation, i.e. to pass from zero order (as in[33]) to first and second order perturbations, and ultimately to tackle the absolute continuity of the elliptic operator

$$H = \sum_{j,l=1}^{d} (D_j - a_j) g_{jl} (D_l - a_l) + V, \quad D_j = -i\partial_j, \tag{1.1}$$

with a periodic variable metric $\{g_{jl}\} = \mathbf{G}$ and a magnetic potential $\mathbf{a} = \{a_l\}$, for arbitrary $d \geq 2$.

The case of first order perturbations, i.e. that of constant $\mathbf{G}$'s and variable $\mathbf{a}$'s, was handled in [15] (small $\mathbf{a}$'s), [6], [7] ($d = 2$) and [30] (arbitrary $d \geq 2$). If the metric is conformal, i.e. the matrix $\mathbf{G}$ is given by a scalar multiple of the identity matrix, then the problem can be easily reduced to the case of a constant metric. This situation is discussed in [8]. The most difficult case, that of a general variable $\mathbf{G}$, remained unaccessible until the paper [25], where it was resolved for infinitely differentiable $\mathbf{G}$'s, $\mathbf{a}$'s and $d = 2$. An important breakthrough was made in the recent work [11] where the absolute continuity

*Date*: 26 July 2001.
2000 *Mathematics Subject Classification.* 35J10, 30C20, 35J25.
This work is supported by the Royal Society.

was proved in all dimensions $d \geq 2$, but with an additional requirement of the reflectional symmetry of the operator. Without the symmetry assumption the question is still open. At present it is only known that without smoothness assumptions on the coefficients the absolute continuity may break down. An appropriate example with $V = 0, \mathbf{a} = 0$ and a "non-smooth" $\mathbf{G}$ was constructed in [10].

Most of the progress was achieved in the two-dimensional case, which we shall discuss in more detail. The paper [25], as well as all the earlier papers on absolute continuity, relied on the approach suggested by L. Thomas in [33]. Later it was observed in [19] that the general periodic metric can be reduced to a conformal (or scalar) periodic one by a suitable isothermal change of variables. This allows one to reproduce the results of [25] even under much weaker assumptions, by reducing the problem to the one considered in [6], [7], [8]. This approach was exploited in [4] for the operator of the form (1.1) with a delta-like periodic potential supported by a periodic system of curves. Even more general perturbations are studied in [27], [28].

The main aim of the present paper is to prove the absolute continuity for the operator of the form (1.1) with non-constant periodic coefficients, defined in a periodic domain $\Omega \subset \mathbb{R}^2$ (which is usually referred to as a waveguide) with the Dirichlet or "natural" boundary conditions. In these two cases we use the notation $H_D$ or $H_N$ for the operator at hand. By the natural boundary conditions we mean either Neumann or the third boundary condition. As in [4], we also include in the operator a delta-like periodic perturbation supported by smooth curves, which allows us to study the cases of the Neumann condition and the third boundary condition simultaneously. In a somewhat restricted generality this problem with Dirichlet and Neumann conditions was considered in [31]. Compared to [31] in the present paper the smoothness conditions on the coefficients are substantially relaxed, and the case of the third boundary-value problem is also treated. The progress has become possible due to a different approach to the problem: instead of Morame's techniques we now use the isothermal coordinate change. A parallel approach is used in the work [29] focused on this problem in a slightly different setting. Similar spectral questions for a periodic Helmholz-type operator were studied in [9]. More general elliptic operators with constant coefficients and a periodic potential perturbation are considered in [18], Theorem 5.4.9. We refer to the book [18], Sect. 5.4 for further relevant references.

The corner stone of our method is an isothermal change of coordinates, which reduces the metric to a conformal one. The new coordinates are given by functions satisfying the Beltrami equation (see Sect 6 below) with a dilatation coefficient $q$ determined by the matrix $\mathbf{G}$ (see (6.2)), i.e. they define a $q$-quasi-conformal mapping. We use a $q$-quasi-conformal change of variables which maps the waveguide $\Omega$ homeomorphically onto a straight strip. The existence of such a transformation follows from a "quasi-conformal" version of Riemann's mapping theorem (see e.g. [17], Ch. 1). Here a key point is that the uniqueness part of Riemann's mapping theorem guarantees certain natural "periodicity" property of the above homeomorphism (see (6.4) below). This ensures that the transformed operators $H_D$ and $H_N$ have periodic coefficients.

To be precise, the quasi-conformal Riemann's theorem alone is not sufficient for our needs. It is also necessary to study the boundary behaviour of the quasi-conformal homeomorphism. Moreover, we allow the boundary of the waveguide $\Omega$ to have corners and inward peaks, which give rise to singularities of the map. For conformal maps this circle of questions is exhaustively studied in the relevant literature (see e.g. [26]), and is usually associated with the names of P. Koebe, O.D. Kellogg and S.E. Warschawski. The extension to the quasi-conformal case is either well known or evident to experts in complex analysis. Nevertheless, we have decided to provide precise statements and complete proofs, since we were unable to find them in the literature in the form readily suitable for our purposes. These results along with the quasi-conformal analogue of Riemann's mapping theorem are collected in Sect. 7.

As was mentioned earlier, the periodic isothermal coordinates were used in [19], [4] to establish the absolute continuity of the operator $H$ acting in $\mathsf{L}^2(\mathbb{R}^2)$, which we denote below by $H_F$. They were constructed in [19] in the following way. Using results from [3], [35] one can define an analytic structure on the two-dimensional torus with the help of local $q$-quasi-conformal coordinates. Then it follows from the theory of Riemann surfaces that integration of the analytic differential on the torus leads to the sought isothermal coordinates in $\mathbb{R}^2$. Note that the above analytic differential exists and is unique up to a constant factor, since the torus is a surface of genus one. The secondary aim of our paper is to give another, more direct proof of the existence of such coordinates. Instead of geometrical considerations of [19] we rely on the well-known fact (see [1], [2], [3], [35]) that there exists a unique $q$-quasi-conformal homeomorphism of $\mathbb{C}$ which preserves the points $0$, $2\pi$ and $\infty$. Similarly to the waveguide case, a crucial observation here is that the uniqueness combined with periodicity of $\mathbf{G}$ automatically implies the required periodicity of the homeomorphism at hand (see Theorem 6.1). Using the isothermal coordinates, as in [19] and [4], we obtain the absolute continuity of the operator $H_F$. Moreover, applying a stronger regularity result for the coordinate change, we are able to relax the smoothness restrictions on $\mathbf{G}$ in comparison with [19], [4]: for instance, under the assumption $\det \mathbf{G} = 1$ the matrix $\mathbf{G}$ does not need to be Hölder-continuous, but only bounded.

After reduction to a conformal metric, the absolute continuity of $H_F$ results immediately from the earlier papers [7], [8]. As to the operators $H_D, H_N$, the isothermal change of variables reduces them to operators of the same type acting on a straight strip in $\mathbb{R}^2$, with a scalar constant $\mathbf{G}$. From this point on we follow the strategy suggested in [31]. It consists in further reduction to an auxiliary operator $H_P$ with *periodic* conditions on the boundary of the strip of the double width. Then a reference to [4] secures the required absolute continuity.

The paper is organised as follows. Section 2 contains some preliminary material and the precise statements of all main results of the paper (Theorems 2.6, 2.9). In Section 3 we make first reductions simplifying the problem. In particular it is shown that it suffices to prove the main results for the case $\det \mathbf{G} = 1$. The important Section 4 is devoted to a detailed description of the isothermal change of variables. Two central theorems of this section (Theorems 4.1 and 4.2) are proved in Sect. 6 after having been translated into the language of the quasi-conformal maps. The proof of the main results is completed in Sect. 5. The necessary information on the quasi-conformal maps and their boundary properties is collected in Sect. 7.

## 2. Main results

2.1. **Notation. Lattices and domains.** Let $\mathbf{e}_1, \mathbf{e}_2$ be the canonical basis in $\mathbb{R}^2$. Along with the standard two-dimensional square lattice $\boldsymbol{\Gamma} = (2\pi\mathbb{Z})^2$, introduce two "one-dimensional" lattices:

$$\boldsymbol{\gamma}_1 = (2\pi\mathbb{Z}) \times \{0\} = \{2\pi n \mathbf{e}_1, n \in \mathbb{Z}\},$$
$$\boldsymbol{\gamma}_2 = \{0\} \times (2\pi\mathbb{Z}) = \{2\pi n \mathbf{e}_2, n \in \mathbb{Z}\}.$$

We say that a function $f$ is $\boldsymbol{\gamma}_j$-periodic (resp. $\boldsymbol{\Gamma}$-periodic), if $f(\mathbf{x} + 2\pi n \mathbf{e}_j) = f(\mathbf{x})$ a.a. $\mathbf{x}$ and all $n \in \mathbb{Z}$ (resp. $f(\mathbf{x} + \boldsymbol{\xi}) = f(\mathbf{x})$ a.a. $\mathbf{x}$ and all $\boldsymbol{\xi} \in \boldsymbol{\Gamma}$).

For any set $\mathcal{F} \subset \mathbb{R}^2$ define its translates as follows:

$$\mathcal{F}^{(\mathbf{n})} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} - 2\pi \mathbf{n} \in \mathcal{F}\}, \ \mathbf{n} \in \mathbb{Z}^2,$$
$$\mathcal{F}^{(n)} = \mathcal{F}^{(\mathbf{n})}, \ \text{with} \ \ \mathbf{n} = (n, 0), \ n \in \mathbb{Z}.$$

We say that the set $\mathcal{F}$ is $\boldsymbol{\Gamma}$-*periodic* if $\mathcal{F} = \mathcal{F}^{(\mathbf{n})}$ for all $\mathbf{n} \in \mathbb{Z}^2$, and that $\mathcal{F}$ is $\boldsymbol{\gamma}_1$-*periodic* if $\mathcal{F} = \mathcal{F}^{(n)}$ for all $n \in \mathbb{Z}$. When it does not lead to a confusion, instead of "$\boldsymbol{\gamma}_1$-" or "$\boldsymbol{\Gamma}$-periodicity" we use the term "periodicity". Similarly we define the periodicity of sets $\mathcal{F}$ on the cylinder

$$\mathcal{C} = \mathbb{R}^2/\boldsymbol{\gamma}_2.$$

Precisely, $\mathcal{F} \subset \mathcal{C}$ is said to be periodic (or $\boldsymbol{\gamma}_1$-periodic) if $\mathcal{F}^{(n)} = \mathcal{F}$ for all $n \in \mathbb{Z}$.

We are going to study periodic operators in two-dimensional domains of three types: on the entire plain $\mathbb{R}^2$, on the cylinder $\mathcal{C}$ or on a periodic domain $\Omega \subset \mathbb{R}^2$. Our model periodic domain will be the straight strip

$$\mathcal{S} = \mathcal{S}_{\mathrm{d}} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < \pi\,\mathrm{d}\}, \ \mathrm{d} > 0. \tag{2.1}$$

Obviously, the cylinder $\mathcal{C}$ can be viewed as the closed strip $\overline{\mathcal{S}_2}$ with identified lower and upper boundaries. In general, we assume that $\Omega$ is as described below:

**Definition 2.1.** We say that a domain $\Omega \subset \mathbb{R}^2$ is *admissible* if there exists a finite collection of bounded domains $\mathcal{E}_j, \ j = 1, 2, \dots, N$ with Lipschitz boundaries such that the set

$$\mathcal{E}_0 = \bigcup_{j=1}^{N} \mathcal{E}_j \tag{2.2}$$

is connected, and $\Omega = \cup_{n \in \mathbb{Z}} \mathcal{E}_0^{(n)}$.

Note that the domain $\Omega$ is automatically $\boldsymbol{\gamma}_1$-periodic and bounded in the direction $\mathbf{e}_2$. We also point out that $\Omega$ satisfies the interior cone condition (see e.g. [24], Sect. 1.3.3), since so do the domains $\mathcal{E}_j, j = 1, 2, \dots, N$. Without loss of generality we **always assume that**

$$\mathcal{E}_0 \supset \{\mathbf{x} \in \Omega : -1 \le x_1 \le 2\pi + 1\}. \tag{2.3}$$

Certainly, the choice of the domains $\mathcal{E}_j$ for a given admissible $\Omega$ is not unique.

As a rule we try to treat all three cases simultaneously, and therefore we use the notation $\Lambda$ either for $\mathbb{R}^2$ or $\mathcal{C}$ or $\Omega$. Identifying the points that differ by a vector of the lattice we define

$$\Upsilon = \begin{cases} \Omega/\boldsymbol{\gamma}_1, & \text{if } \Lambda = \Omega; \\ \mathbb{T}^2 = \mathbb{R}^2/\boldsymbol{\Gamma}, & \text{if } \Lambda = \mathbb{R}^2 \text{ or } \mathcal{C}. \end{cases} \tag{2.4}$$

Introduce also the *fundamental domains*:

$$\mathcal{O} = \begin{cases} (0, 2\pi) \times (0, 2\pi), & \text{if } \Lambda = \mathbb{R}^2 \text{ or } \mathcal{C}, \\ \{\mathbf{x} \in \Omega : 0 < x_1 < 2\pi\}, & \text{if } \Lambda = \Omega. \end{cases}$$

In the case $\Lambda = \Omega$ the set $\mathcal{O}$ might not be connected.

Similarly to $\Omega$, for our purposes it will be necessary to view $\mathbb{R}^2$ as being covered by bounded domains. Precisely, we assume that there are finitely many bounded domains $\mathcal{E}_j, j = 1, 2, \dots, N$, with Lipschitz boundaries such that

$$\overline{\mathcal{O}} \subset \mathcal{E}_0 = \bigcup_{j=1}^{N} \mathcal{E}_j.$$

**System of curves.** We also need to introduce a system of curves $\boldsymbol{\Sigma}$ in $\Lambda$ associated with the covering of $\Lambda$ by $\mathcal{E}_j$'s and their translates. Below by a "$\mathsf{C}^{m+\alpha}$-arc", $m \in \mathbb{N}$, $0 \le \alpha < 1$, we mean a Jordan arc in $\mathbb{R}^2$ which is parametrised by a $\mathbf{C}^{m+\alpha}$-smooth function $\boldsymbol{\psi} : [0, 1] \to \mathbb{R}^2$ such that $|\boldsymbol{\psi}'(t)| > 0,\ t \in [0, 1]$.

**Definition 2.2.** Let $\Lambda = \mathbb{R}^2$ or $\Omega$, and let the parameter $\mathbf{n}$ vary over the set $\mathbb{Z}^2$ ( for $\Lambda = \mathbb{R}^2$) or $\mathbb{Z} \times \{0\}$ (for $\Lambda = \Omega$). Let $\ell_j \subset \overline{\mathcal{E}_j},\ j = 1, 2, \dots N$, be a finite set of $\mathsf{C}^1$-arcs such that

$$\ell_j^{(\mathbf{n})} \cap \ell_j = \emptyset,\ \forall \mathbf{n} \ne 0.$$

Then *a (periodic) system of curves* $\boldsymbol{\Sigma}$ *in* $\Lambda$ is defined to be a *family* $\{\Sigma_j\}$ of the closed sets $\Sigma_j \subset \overline{\Lambda}$ of the form

$$\Sigma_j = \cup_{\mathbf{n}} \ell_j^{(\mathbf{n})}. \tag{2.5}$$

Let $\Lambda = \mathcal{C}$. Then the set $\boldsymbol{\Sigma}$ is *a system of curves in* $\mathcal{C}$ if there exists a system of curves $\mathbf{M}$ in $\mathbb{R}^2$ such that $\boldsymbol{\Sigma} = \mathbf{M}/\boldsymbol{\gamma}_2$, with the latter being defined as the family $\{\mathrm{M}_j/\boldsymbol{\gamma}_2\}$.

Similarly one defines

$$\boldsymbol\Xi = \begin{cases} \boldsymbol\Sigma/\boldsymbol\gamma_1, & \text{if } \ \Lambda = \Omega \ \text{ or } \ \mathcal{C}; \\ \boldsymbol\Sigma/\boldsymbol\Gamma, & \text{if } \ \Lambda = \mathbb{R}^2, \end{cases} \qquad j = 1, 2, \dots, N. \tag{2.6}$$

*Remark* 2.3. (i) In the above definition any two curves $\ell_j$, $\ell_l$ or their parts can coincide. This means in particular that $\boldsymbol\Sigma$ may contain several "copies" of the same curve. The curves are also allowed to meet at zero angle.

(ii) Definition 2.2 prescribes exactly one curve $\ell_j$ for each of the domains $\mathcal{E}_j$. A seemingly more general situation when there are finitely many curves in $\overline{\mathcal{E}_j}$ is easily reduced to the original one by adding a necessary number of copies of $\mathcal{E}_j$ in (2.2).

For any bounded set $\mathcal{E} \subset \overline{\Lambda}$ we denote

$$\boldsymbol\Sigma_{\mathcal{E}} = \{\Sigma_{j,\mathcal{E}}\}, \quad \Sigma_{j,\mathcal{E}} = \cup_{\mathbf{n}} \ell_j^{(\mathbf{n})}, \tag{2.7}$$

where $\mathbf{n}$ are such that $\ell_j^{(\mathbf{n})} \cap \mathcal{E} \neq \emptyset$. Obviously, each $\Sigma_{j,\mathcal{E}}$ contains finitely many translates of $\ell_j$.

**Function spaces.** Apart from the standard notational conventions for classes of differentiable functions we use the notation $\mathsf{W}^{s,p}(\mathcal{C})$, $s = 0, 1$, $p \geq 1$ which stands for the space of all $\boldsymbol\gamma_2$-periodic functions from $\mathsf{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^2)$ equipped with the norm $\| \cdot \|_{\mathsf{W}^{s,p}(\mathcal{S}_2)}$. Similarly, the notation $\mathsf{W}^{s,p}(\Upsilon)$, $s = 0, 1$, $p \geq 1$, means the space of all $\boldsymbol\gamma_1$-periodic ( for $\Lambda = \Omega$) or $\boldsymbol\Gamma$-periodic (for $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathcal{C}$) functions $u \in \mathsf{W}^{s,p}_{\mathrm{loc}}(\Lambda)$ with the norm $\| \cdot \|_{\mathsf{W}^{s,p}(\mathcal{O})}$. In the case $p = 2$ we use the standard notation $\mathsf{H}^s \equiv \mathsf{W}^{s,2}$. The same convention applies to Hölder spaces $\mathsf{C}^{m+\alpha}$. For instance $\mathsf{C}^{m+\alpha}(\Upsilon)$, $m \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1)$ (resp. $\mathsf{C}^{m+\alpha}(\overline{\Upsilon})$) denotes the space of all $\boldsymbol\gamma_1$-periodic ( for $\Lambda = \Omega$) or $\boldsymbol\Gamma$-periodic (for $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathcal{C}$) functions $u \in \mathsf{C}^{m+\alpha}(\Lambda)$ (resp. $\mathsf{C}^{m+\alpha}(\overline{\Lambda})$ ). Certainly, in the case $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathcal{C}$ the space $\mathsf{C}^{m+\alpha}(\Upsilon)$ coincides with $\mathsf{C}^{m+\alpha}(\overline{\Upsilon})$.

To denote spaces of vector-valued functions, we use boldface letters, e.g. $\mathbf{L}^p(\Lambda)$. The notation $\mathfrak{D}\mathbf{f}$ stands for the Jacobian matrix of the function $\mathbf{f}$. Slightly abusing the notation we sometimes do not distinguish between $\mathsf{L}^2(\mathcal{C})$ and $\mathsf{L}^2(\mathcal{S}_2)$ ($\mathsf{L}^2(\Upsilon)$ and $\mathsf{L}^2(\mathcal{O})$).

Function spaces on $\boldsymbol\Sigma$ are defined in a natural way. Namely, by definition the space $\mathbf{L}^p(\boldsymbol\Sigma)$ is a set of functions $\boldsymbol\sigma = \{\sigma_j\}$ such that $\sigma_j \in \mathsf{L}^p(\Sigma_j), j = 1, 2, \dots, N$. Similarly one defines $\mathbf{L}^p(\boldsymbol\Xi)$. We say that $\boldsymbol\sigma$ is real-valued if all the components $\sigma_j$ are real-valued.

Definition of traces on $\boldsymbol\Sigma$ of functions on $\Lambda$ requires special comments. Suppose that $f \in \mathsf{H}^1(\Lambda)$. The trace $f|_{\ell_j}$ is defined to be the trace of the function $f|_{\mathcal{E}_j} \in \mathsf{H}^1(\mathcal{E}_j)$. Sometimes we write $f$ instead of the trace $f|_{\ell_j}$ when it does not cause confusion. By the embedding theorems and multiplicative inequality for the traces (see [23], Corollary 1.4.7/2), $f \in \mathsf{L}^r(\ell_j)$ with any $r < \infty$ and

$$\|f\|_{\mathsf{L}^r(\ell_j)} \leq \epsilon \|\nabla u\|_{\mathsf{L}^2(\mathcal{E}_j)} + C_j(\epsilon) \|u\|_{\mathsf{L}^2(\mathcal{E}_j)}, \tag{2.8}$$

for all $\epsilon > 0$. Similarly one defines the traces on the translates $\ell_j^{(\mathbf{n})}$, which leads to the collection of traces $f|_{\boldsymbol\Sigma} = \{f|_{\Sigma_j}\}$ in a natural way. Note that if an arc $\ell = \ell_j = \ell_m$ belongs to $\mathcal{E}_j \cap \mathcal{E}_m$, then $f|_{\ell_j} = f|_{\ell_m}$. On the contrary, if $\ell \subset \partial\mathcal{E}_j \cap \partial\mathcal{E}_m$ for two distinct $j$ and $m$, then $f|_{\ell_j}$ may be different from $f|_{\ell_m}$. The latter situation can occur in the case when $\Lambda = \Omega$ and $\ell$ is a part of the boundary of $\Omega$ such that $\Omega$ lies on both sides of $\ell$.

**Weights and coefficients.** We work in the weighted space $\mathsf{L}^2(\mu, \Lambda)$ with the norm

$$\|u\|_\mu = \left[\int_\Lambda |u|^2 \mu d\mathbf{x}\right]^{1/2},$$

where $\mu$ is a real-valued periodic function, satisfying the conditions

$$\begin{cases} \operatorname{mes}\{\mathbf{x} \in \Lambda : \mu(\mathbf{x}) \leq 0\} = 0, \\ \mu \in \mathsf{L}^t(\Upsilon),\ t > 1. \end{cases} \tag{2.9}$$

In the cases when it does not cause any confusion, the subscript $\mu$ will be omitted from the notation of the norm. We are interested in the properties of the Schrödinger type periodic operators in $\mathsf{L}^2(\mu, \Lambda)$, defined by the formal expression

$$H = \frac{1}{\mu}\langle(\mathbf{D} - \mathbf{a}), \omega^2 \mathbf{G}(\mathbf{D} - \mathbf{a})\rangle + \frac{1}{\mu}V + \boldsymbol{\sigma}\delta_{\boldsymbol{\Sigma}},\ \mathbf{D} = -i\nabla,$$

where $V, \omega, \mathbf{a}, \mathbf{G}$ are some real-valued periodic (vector/matrix) functions defined on $\Lambda$, and $\boldsymbol{\sigma}$ is a real-valued function defined on a system of curves $\boldsymbol{\Sigma}$. Let us now give a precise definition of the operators in question. The *electric potential* $V$, the *magnetic vector-potential* $\mathbf{a}$ and the function $\boldsymbol{\sigma}$ are supposed to satisfy the conditions

$$\mathbf{a} \in \mathbf{L}^s(\Upsilon),\ s > 2;\quad V \in \mathsf{L}^p(\Upsilon),\ p > 1, \tag{2.10}$$

$$\boldsymbol{\sigma} \in \mathbf{L}^r(\Xi),\ r > 1. \tag{2.11}$$

The coefficient $\mathbf{G} = \{g_{jl}(\mathbf{x})\}, j, l = 1, 2$ is a symmetric matrix-valued function on $\Upsilon$ with real-valued entries $g_{jl}(\mathbf{x})$ that satisfy the condition

$$\begin{cases} c|\boldsymbol{\xi}|^2 \leq \langle \mathbf{G}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \leq C|\boldsymbol{\xi}|^2, \\ \det \mathbf{G}(\mathbf{x}) = C', \end{cases} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^2,\ \text{a.a.}\ \ \mathbf{x} \in \Upsilon. \tag{2.12}$$

Here and everywhere below by $C$ and $c$ with or without indices we denote various positive constants whose precise value is unimportant. As a rule we assume that $\det \mathbf{G} = 1$, but sometimes it is convenient not to have this restrictions.

As to $\omega$, it is a real-valued function on $\Upsilon$, such that

$$c \leq \omega(\mathbf{x}) \leq C,\quad \text{a.a.}\ \ \mathbf{x} \in \Upsilon. \tag{2.13}$$

We are interested in four different realisations of the operator $H$. Namely, we study

- $H$ as an operator in $\mathsf{L}^2(\mu, \mathbb{R}^2)$, which will be later referred to as the "full" operator $H_F$;
- $H$ as an operator in $\mathsf{L}^2(\mu, \mathcal{C})$, or which is the same, as an operator in $\mathsf{L}^2(\mu, \mathcal{S}_2)$ with periodic conditions on the boundary of $\mathcal{S}_2$. In this case we use the notation $H_P$;

- $H$ acting in $\mathsf{L}^2(\mu, \Omega)$ with the Dirichlet or natural boundary condition. In these cases we use the notation $H_D$ or $H_N$ respectively.

When we need to treat all four situations simultaneously, we use the notation $H_\aleph$, where $\aleph$ will have the meaning of any of the four letters $F, P, D, N$.

For each of these four problems the operator will be defined via its quadratic form. To give this definition it suffices to assume (2.9), (2.10), (2.11), (2.12) and (2.13), although later on we shall need more restrictive conditions on $\mathbf{G}$ and $\omega$. Consider the following quadratic form

$$h[u] = \int_\Lambda \omega^2 \langle \mathbf{G}(\mathbf{D} - \mathbf{a})u, \overline{(\mathbf{D} - \mathbf{a})u}\rangle d\mathbf{x} + \int_\Lambda V|u|^2 d\mathbf{x} + \int_{\mathbf{\Sigma}} \boldsymbol{\sigma}|u|^2 dS, \tag{2.14}$$

where

$$\int_{\mathbf{\Sigma}} \boldsymbol{\sigma}|u|^2 dS = \sum_{j=1}^N \int_{\Sigma_j} \sigma_j \big|u|_{\Sigma_j}\big|^2 dS \tag{2.15}$$

defined either on $\mathcal{D}_F = \mathsf{H}^1(\mathbb{R}^2)$ (for the full problem), or $\mathcal{D}_P = \mathsf{H}^1(\mathcal{C})$ (for the $\boldsymbol{\gamma}_2$-periodic problem), or $\mathcal{D}_D = \mathsf{H}^1_0(\Omega)$ or $\mathcal{D}_N = \mathsf{H}^1(\Omega)$. Depending on the domain we denote the form $h$ by $h_F, h_P, h_D$ or $h_N$ respectively. Sometimes, in order to distinguish operators defined on different domains or/and with different coefficients and weights, we use the full notation $H_\aleph(\omega, \mathbf{G}, \mathbf{a}, V, \mu, \boldsymbol{\sigma}; \Lambda)$ or its short variants: e.g. $H_\aleph(\omega, \mathbf{G})$. Similar convention applies to the notation of the quadratic forms $h_\aleph$.

Let us check that these forms are closed in $\mathsf{L}^2(\mu, \Lambda)$. To this end split the form (2.14) into the *unperturbed* form and the *perturbation* form:

$$\begin{aligned} h_\aleph[u] &= h^{(0)}_\aleph[u] + w_\aleph[u]; \\ h^{(0)}_\aleph[u] &= \int_\Lambda \omega^2 \langle \mathbf{G}\mathbf{D}u, \overline{\mathbf{D}u}\rangle d\mathbf{x}, \\ w_\aleph[u] &= \int_\Lambda \big[-\omega^2\langle \mathbf{G}\mathbf{D}u, \mathbf{a}\rangle \overline{u} - \omega^2 u \langle \mathbf{G}\mathbf{a}, \overline{\mathbf{D}u}\rangle + \omega^2 \langle \mathbf{G}\mathbf{a}, \mathbf{a}\rangle |u|^2\big] d\mathbf{x} \\ &\qquad + \int_\Lambda V|u|^2 d\mathbf{x} + \int_{\mathbf{\Sigma}} \boldsymbol{\sigma}|u|^2 dS. \end{aligned}$$

The necessary properties of these forms are contained in

**Proposition 2.4.** *Let $\mu$ satisfy the condition* (2.9) *and let $\mathbf{a}$, $V$ and $\boldsymbol{\sigma}$ satisfy* (2.10), (2.11)*. Then*

(i) *The standard $\mathsf{H}^1$-norm is equivalent to the norm induced by the form $h^{(0)}_\aleph$, i.e.*

$$C^{-1}(\|\nabla u\|_1^2 + \|u\|_1^2) \le h^{(0)}_\aleph[u] + \|u\|^2_\mu \le C(\|\nabla u\|_1^2 + \|u\|_1^2),\ \forall u \in \mathcal{D}_\aleph, \tag{2.16}$$

*with some constant $C$ depending on $\omega, \mathbf{G}, \mu$;*

(ii) *The forms $h^{(0)}_\aleph$ with the domains $\mathcal{D}_\aleph$ are closed in $\mathsf{L}^2(\mu, \Lambda)$;*

(iii) *For any* $\epsilon > 0$ *there exists a constant* $C_\epsilon$ *such that*

$$|w_{\aleph}[u]| \le \epsilon h^{(0)}_{\aleph}[u] + C_\epsilon \|u\|^2_\mu, \ \forall u \in \mathcal{D}_{\aleph}, \tag{2.17}$$

*so that the perturbed forms* $h_\aleph$ *are also closed.*

The plan of the proof is to establish the required estimates on the domain $\mathcal{E}_0$ using the embedding theorems for the domains with the interior cone condition, and the estimate (2.8) for the traces. The bounds for the entire $\Lambda$ are obtained by using the periodicity of $\Lambda$ and an appropriate partition of unity. The details are fairly standard and are omitted.

By Proposition 2.4 all four forms $h_\aleph$ are closed, and therefore they uniquely define in $\mathsf{L}^2(\mu, \Lambda)$ four self-adjoint operators which we denote by $H_F$, $H_P, H_D$ and $H_N$. We do not need to know the domains of these operators, although they can be specified under supplementary regularity conditions on the coefficients, the boundary of $\Omega$ and the curves from $\mathbf{\Sigma}$. For instance, if the system of curves contains only the boundary $\partial\Omega$, then $H_N$ is the operator of the third boundary value problem with the condition

$$\omega^2 \langle \mathbf{G}(\nabla - i\mathbf{a})u, \mathbf{n}\rangle + \sigma u = 0, \ \mathbf{x} \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the exterior unit normal to the boundary at $\mathbf{x} \in \partial\Omega$. If $\mathbf{\Sigma}$ contains a curve which has an arc $\ell$ strictly inside $\Omega$ and separated from other components of $\mathbf{\Sigma}$, then the integral over $\mathbf{\Sigma}$ in (2.14) induces the condition

$$\big[\omega^2 \langle \mathbf{G}\nabla u, \mathbf{n}\rangle\big] + \sigma u = 0, \ \mathbf{x} \in \ell,$$

on the jump $[\dots]$ of the conormal derivative across the curve $\ell$.

An important role will be played by the general observation that the singular continuous spectra of the operators $H_\aleph$ are empty, which we state separately for later reference:

**Proposition 2.5.** *Let* $\Lambda$ *be either* $\mathbb{R}^2$ *or* $\mathcal{C}$ *or* $\Omega$. *Suppose that the conditions* (2.9), (2.10), (2.12), (2.13) *and* (2.11) *are fulfilled. Then the singular continuous spectra of* $H_\aleph$ *are empty.*

The proof of this property is based on the standard direct integral representation for $H_\aleph$, known as *the Floquet decomposition*. The crucial fact is that the resolvents of the fibers of $H_\aleph$ in this representation are compact operator-functions, analytic in the quasi-momentum. We are not going to provide all the details of this argument, but refer to the comprehensive exposition of this issue in [18], and also to [19], [15] and [36]. The Floquet decomposition for $H_F$ can be found, for example, in [4] and [8]. For the operators $H_D, H_N$ it is clearly explained in [5].

2.2. **Results.** Our main goal is to go further and prove that the spectrum of $H_\aleph$ is absolutely continuous. Here $\aleph$ takes the values $F, D$ or $N$. The operator $H_P$ plays an auxiliary role: we prove its absolute continuity only in the case of a diagonal constant matrix $G$. This result will be decisive in the proofs for the cases $D$ and $N$.

Now the conditions (2.12), (2.13) are insufficient. Suppose in addition that

$$\omega \in \mathsf{W}^{1,q}(\Upsilon) \ \text{ and } \ \mathbf{G}\nabla\omega \in \mathbf{W}^{1,q}(\Upsilon), \ q > 1. \tag{2.18}$$

As far as $\mathbf{G}$ is concerned, a number of results will be obtained under the additional restriction

$$\mathbf{G} \in \mathsf{C}^\alpha\big(\overline{\Upsilon}\big), \quad \alpha \in (0,1). \tag{2.19}$$

Clearly, for a uniformly Lipshitz matrix $\mathbf{G}$ the condition (2.18) is equivalent to $\omega \in \mathsf{W}^{2,q}, q > 1$. One is tempted to say that due to the presence of the function $\omega$ in (2.14), the condition $\det \mathbf{G} = \text{const}$ in (2.12) does not restrict generality. We emphasise however that the smoothness conditions for the functions $\mathbf{G}$ and $\omega$ are different.

The main result for the operator $H_F$ is contained in the next theorem.

**Theorem 2.6.** *Let $\Lambda = \mathbb{R}^2$, and $\boldsymbol{\sigma} = 0$. Suppose that the conditions* (2.9), (2.10), (2.12), (2.13) *and* (2.18) *are fulfilled. Then the spectrum of $H_F$ is absolutely continuous.*

*Remark* 2.7. For a variable $\mathbf{G}$ satisfying (2.19) Theorem 2.6 was proved in [4] even with a $\boldsymbol{\sigma} \neq 0$. R. Shterenberg [28] recently proved the absolute continuity of $H_F$ under the reduced smoothness assumption $\omega \in \mathsf{W}^{1,q}$, $q > 2$. For our proof we need an earlier result from the paper [7] where Theorem 2.6 was established for constant matrices $\mathbf{G}$ and $\omega = 1$.

To prove the absolute continuity of the operators $H_D$, $H_N$ we need to impose some extra conditions on the domain $\Omega$. We assume that $\Omega$ is $\boldsymbol{\gamma}_1$-periodic and that there exists a homeomorphism of $\overline{\Omega}$ onto $\overline{\mathbb{S}_1}$. This means, in particular, that the boundary of $\Omega$ consists of two disjoint $\boldsymbol{\gamma}_1$-periodic Jordan curves that we denote by $\ell_-, \ell_+$. We impose the following condition on these curves:

**Condition 2.8.** (1) *Locally, each curve $\ell_-, \ell_+$, is a piece-wise $\mathsf{C}^{1+\alpha}$-smooth Jordan arc with $\alpha \in (0,1)$;*
(2) *The domain $\Omega$ does not have any outward peaks, i.e. the interior angle between any two smooth components of the boundary at each point of non-smoothness is strictly greater than zero.*

Note that Condition 2.8 does not exclude *inward* peaks, and that the cone condition is satisfied. Using this fact and remembering that every bounded domain with cone condition can be represented as a union of finitely many domains with Lipshitz boundaries (see e.g. [23]), one can easily show that the domain $\Omega$ is admissible in the sense of Definition 2.1.

The parameter $\alpha$ above, without loss of generality can be chosen the same as in the condition (2.19). Below we denote by $\mathbf{n}(\mathbf{x})$, a.a. $\mathbf{x} \in \partial\Omega$, the exterior unit normal to the boundary $\partial\Omega$, and by $Z \subset \partial\Omega$ the discrete set where the smoothness of the boundary breaks down. Clearly, $Z$ has no finite accumulation points.

**Theorem 2.9.** *Let $\Lambda = \Omega$. Suppose that the domain $\Omega$ satisfies Condition 2.8, the set $\boldsymbol{\Sigma}$ is a system of curves on $\Omega$ in the sense of Definition 2.2, and the conditions* (2.9), (2.10), (2.12), (2.13), (2.11), (2.18), (2.19) *are fulfilled. Then the spectra of $H_D$ and $H_N$ are absolutely continuous.*

For later convenience we make a couple of simplifying assumptions that do not restrict generality.

First we include the boundary $\partial\Omega$ in the system of curves $\boldsymbol\Sigma$, even if $\boldsymbol\Sigma$ already contains either pieces of $\partial\Omega$ or the entire boundary. The notation for the components of $\boldsymbol\Sigma$ will be as follows. By Condition 2.8 there are finitely many $\mathsf{C}^{1+\alpha}$-arcs $\ell_j \subset \partial\Omega,\ j = 1, 2, \dots, M < \infty$ such that

$$\partial\Omega = \cup_{j=1}^{M} \Sigma_j,$$

where $\Sigma_j$ are defined by (2.5). Since the boundary $\partial\Omega$ is a Jordan curve, we can assume that any pair of arcs $\ell_j, \ell_s$ with $s, j = 1, 2, \dots, M$ do not have common interior points. Moreover, since the outward peaks are absent, there are $M$ bounded domains $\mathcal{E}_j \subset \Omega$ with Lipschitz boundaries such that $\ell_j \subset \overline{\mathcal{E}_j}$. This allows us to include $\Sigma_j, j = 1, 2, \dots, M$ in the initial system of curves in $\Omega$. From now on we consider the sets $\Sigma_j, j = 1, 2, \dots, M$ to be the first $M$ components of $\boldsymbol\Sigma$, and the original components of $\boldsymbol\Sigma$ will be relabelled to have numbers from $M + 1$ to $N$. Also we set $\sigma_j = 0,\ j = 1, 2, \dots, M$. Obviously, this procedure does not change the operators $H_D, H_N$, since the quadratic form (2.14) remains unchanged.

Secondly, we assume that if a curve $\ell_j,\ j = 1, 2, \dots, N$, contains a point $\mathbf{z} \in Z$, then $\mathbf{z}$ is either the start or the end point of $\ell_j$, i.e. given a parametrisation $\boldsymbol\psi_j : [0, 1] \to \ell_j$, we have $\boldsymbol\psi_j(0) = \mathbf{z}$ or $\boldsymbol\psi_j(1) = \mathbf{z}$. This can be done by breaking, if necessary, every $\ell_j$, containing a $\mathbf{z} \in Z$, into subarcs, and using Remark 2.3(ii).

## 3. Preliminary conclusions

3.1. **Reduction to $\mu = \omega = 1$.** Proposition 2.5 allows one to show that it suffices to prove Theorems 2.6, 2.9 for $\omega = \mu = 1$. The following Lemma is a variant of a well-known result (see e.g. [8]), and it is a crucial ingredient in our argument.

**Lemma 3.1.** *In addition to the conditions of Proposition 2.5 assume also that* (2.18) *is satisfied. In the case of the operator $H_N$ assume also that Condition 2.8 is fulfilled. Then*

$$\omega^{-1} H_\aleph(\omega, V, \boldsymbol\sigma)\omega^{-1} = H_\aleph(1, \tilde V, \tilde{\boldsymbol\sigma}), \tag{3.1}$$

$$\tilde V = \omega^{-2} V + \omega^{-1}\langle\nabla, \mathbf{G}\nabla\rangle\omega,$$

$$\tilde\sigma_j = \begin{cases} -\omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle, & \text{if } \ j = 1, 2, \dots, M, \\ \omega^{-2}\sigma_j, & \text{if } \ j = M + 1, \dots, N. \end{cases}$$

*Proof.* We prove the Lemma for the case $\aleph = N$ only. To avoid cumbersome calculations assume that $\mathbf{a} = 0, V = 0, \boldsymbol\sigma = 0$. The general case requires only obvious modifications. The second condition in (2.18) and (2.12) imply that $\nabla\omega \in \mathsf{L}^h(\Upsilon)$ with some $h > 2$. Therefore $\omega \in \mathsf{W}^{1,h}(\Upsilon) \subset \mathsf{C}(\Upsilon)$, and using (2.13), one can show that the functions $\omega$, $\omega^{-1}$ are multipliers in $\mathsf{H}^1(\Omega)$. This implies that the quadratic forms of the operators in the r.h.s. and l.h.s. of (3.1) are both closed on $\mathsf{H}^1(\Omega)$. Thus it suffices to prove that the corresponding bilinear forms coincide. Let us consider the form of the operator in the l.h.s.

for $u, v \in \mathsf{H}^1(\Omega)$ (below all integrals are over $\Omega$ unless indicated otherwise):

$$h_N^{(0)}(\omega)[\omega^{-1}u, \omega^{-1}v] = \int \omega^2 \langle \mathbf{G}\nabla(\omega^{-1}u), \overline{\nabla(\omega^{-1}v)}\rangle d\mathbf{x}$$

$$= \int \omega^2 \langle \mathbf{G}(\omega^{-1}\nabla u - \omega^{-2}u\nabla\omega), \omega^{-1}\overline{\nabla v} - \omega^{-2}\overline{v}\nabla\omega\rangle d\mathbf{x}$$

$$= \int \langle \mathbf{G}\nabla u, \overline{\nabla v}\rangle d\mathbf{x} + \int \omega^{-2}\langle \mathbf{G}\nabla\omega, \nabla\omega\rangle u\overline{v} d\mathbf{x}$$

$$- \int \omega^{-1} u \langle \mathbf{G}\nabla\omega, \overline{\nabla v}\rangle d\mathbf{x} - \int \omega^{-1}\langle \mathbf{G}\nabla u, \nabla\omega\rangle \overline{v} d\mathbf{x}.$$

Integrate the last integral by parts, remembering that $\mathbf{G}\nabla\omega \in \mathbf{W}^{1,q}, q > 1$, and using the notation $\mathbf{n}(\mathbf{x})$ for the exterior unit normal to the boundary $\partial\Omega$ at the point $\mathbf{x}$:

$$- \int \omega^{-1}\langle \mathbf{G}\nabla u, \nabla\omega\rangle \overline{v} d\mathbf{x} = - \int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\overline{v} dS$$

$$+ \int \omega^{-1} u \langle \mathbf{G}\nabla\omega, \overline{\nabla v}\rangle d\mathbf{x} + \int \omega^{-1} u\overline{v} \langle \nabla, \mathbf{G}\nabla\rangle \omega d\mathbf{x}$$

$$- \int \omega^{-2}\langle \mathbf{G}\nabla\omega, \nabla\omega\rangle u\overline{v} d\mathbf{x}.$$

Substituting this in the initial formula for the bilinear form, we arrive at the relation

$$h_N^{(0)}(\omega)[\omega^{-1}u, \omega^{-1}v] = h_N^{(0)}(1)[u, v] + \int \tilde{V} u\overline{v} d\mathbf{x} - \int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\overline{v} dS$$

$$= h_N(1, \tilde{V}, \tilde{\boldsymbol{\sigma}})[u, v].$$

It remains to notice that

$$\int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\overline{v} dS = \sum_{j=1}^{M} \int_{\Sigma_j} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\overline{v} dS,$$

which completes the proof. □

**Corollary 3.2.** *It suffices to prove Theorems 2.6 and 2.9 for $\omega = 1$ and $\mu = 1$.*

*Proof.* To be definite, consider the case of Theorem 2.9 only. Suppose that it holds for $\omega = \mu = 1$. First we show that the operator $H_\aleph(1, V, \mu)$, $\aleph = D, N$, is absolutely continuous if $\mu$ satisfies (2.9). According to Proposition 2.5 it suffices to check that it has no point spectrum. For a contradiction, suppose that $\lambda$ is an eigenvalue of $H_\aleph$ with an eigenfunction $u$. Recall that by Proposition 2.4 the quadratic form $h_\aleph$ is closed on the domain $\mathcal{D}_\aleph$ *independent* of $\mu$. Therefore the equality

$$h_\aleph(1, V, \mu, \boldsymbol{\sigma})[u, v] - \lambda \int_\Omega u\overline{v}\mu d\mathbf{x} = 0, \forall v \in \mathcal{D}_\aleph,$$

implies that the point $\tilde\lambda = 0$ is an eigenvalue of the operator $\tilde H_\aleph = H_\aleph(1, V - \lambda\mu, 1, \boldsymbol\sigma)$. On the other hand, the potential $V - \lambda\mu$ satisfies (2.10), so that by Theorem 2.9 with $\omega = \mu = 1$ the spectrum of $\tilde H_\aleph$ is absolutely continuous. This contradiction proves the claim.

Let us now remove the condition $\omega = 1$. Again, in view of Proposition 2.5 it suffices to show that the operator $H_\aleph(\omega, V, \mu, \boldsymbol\sigma)$ has no point spectrum. Assuming the contrary, we obtain from (3.1) that for any eigenfunction $u$ of $H_\aleph(\omega, V, \mu, \boldsymbol\sigma)$ associated with an eigenvalue $\lambda$, the function $\omega u$ will be an eigenfunction of the operator $\tilde H_\aleph = H_\aleph(1, \tilde V - \lambda\omega^{-2}, \mu, \tilde{\boldsymbol\sigma})$ associated with the eigenvalue $\tilde\lambda = 0$. This contradicts the absolute continuity of $\tilde H_\aleph$, which follows from Theorem 2.9 with arbitrary $\mu$ and $\omega = 1$, in the same way as in the first part of the proof. □

Referring to this Corollary, from now on **we always assume that** $\omega = \mu = 1$.

3.2. **Absolute continuity.** The proof of Theorem 2.9 will be based on the further reduction to the operator $H_P(1, \mathbf{B}, \mathbf{a}, V, 1, \boldsymbol\sigma; \mathcal{C})$ with a constant *diagonal* matrix $\mathbf{B}$:

**Proposition 3.3.** *Let $\omega = \mu = 1$. Suppose that the conditions* (2.10), (2.11) *are fulfilled and that $\mathbf{G} = \mathbf{B}$ is a constant* ***diagonal*** *matrix with positive entries. Then the operator $H_P(1, \mathbf{B}, \mathbf{a}, V, 1, \boldsymbol\sigma; \mathcal{C})$ is absolutely continuous.*

Although this result is not contained in [4], it follows immediately from the estimates obtained in [4] and we do not comment on the details.

## 4. Isothermal coordinates. Proof of Theorem 2.6

As it was already explained, the proofs of Theorems 2.6 and 2.9 are based on a reduction of the operator $H_\aleph(\mathbf{G})$ to the canonical form, i.e. to the operator $H_\aleph(\mathbf{A})$ with a constant positive-definite matrix $\mathbf{A}$ (Recall again that we may assume without loss of generality that $\omega = \mu = 1$). This reduction is done using the so-called isothermal coordinates. The required properties of this coordinate change are stated in Theorems 4.1 and 4.2. Their proof is postponed until Section 6.

We consider two mappings. One is a homeomorphism of the entire plane onto itself, and the other is a homeomorphism of the periodic domain $\Omega$ onto the straight strip $\mathcal{S}_{\mathrm{d}}$.

We always denote $\mathbf{F} = \sqrt{\mathbf{G}}$ and assume that

$$\det \mathbf{G} = 1, \tag{4.1}$$

so that $\det \mathbf{F} = 1$ as well.

4.1. **Change of variables.** Below we denote

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The first theorem below describes a suitable coordinate change in $\mathbb{R}^2$.

**Theorem 4.1.** *Let* $\mathbf{G}$ *satisfy* (2.12) *and* (4.1). *Then there exists a unique homeomorphism* $\mathbf{f} = \mathbf{f}_{\mathbb{R}^2} = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$ *such that*

(i) $\mathbf{f}(0) = 0$, $\mathbf{f}(2\pi\mathbf{e}_1) = 2\pi\mathbf{e}_1$, *and* $|\mathbf{f}(\mathbf{x})| \to \infty$ *as* $|\mathbf{x}| \to \infty$;

(ii) *The components* $f_1, f_2$ *satisfy the equation*

$$\nabla f_2 = \mathbf{J}\mathbf{G}\nabla f_1, \ \text{a.a. } \mathbf{x} \in \mathbb{R}^2. \tag{4.2}$$

*Moreover, the map* $\mathbf{f}$ *possesses the following properties:*

(iii) *The Jacobian* $J_{\mathbf{f}}(\mathbf{x}) = \det\big(\mathfrak{D}\mathbf{f}(\mathbf{x})\big)$ *is positive a.a.* $\mathbf{x} \in \mathbb{R}^2$. *The function* $\mathbf{f}$ *and its inverse* $\mathbf{f}^{-1}$ *both belong to* $\mathbf{W}^{1,\tau}_{\mathrm{loc}}(\mathbb{R}^2)$ *with some* $\tau > 2$.

(iv) *For any* $u \in \mathsf{H}^1_{\mathrm{loc}}$ *the usual chain rule holds:*

$$\nabla(u \circ \mathbf{f}) = [\mathfrak{D}\mathbf{f}]^T(\nabla u \circ \mathbf{f}), \quad \text{a.a. } \ \mathbf{x} \in \mathbb{R}^2.$$

*Moreover, for any* $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$ *the function* $h \circ \mathbf{f}$ *belongs to* $\mathsf{L}^1_{\mathrm{loc}}(J_{\mathbf{f}}, \mathbb{R}^2)$ *and*

$$\int_{\mathbf{f}(\Omega_1)} h(\mathbf{y})d\mathbf{y} = \int_{\Omega_1} \big(h \circ \mathbf{f}\big)(\mathbf{x}) J_{\mathbf{f}}(\mathbf{x}) d\mathbf{x}$$

*for any open bounded* $\Omega_1 \subset \mathbb{R}^2$.

(v) *For* $\mathbf{h}_1 = 2\pi\mathbf{e}_1$ *and some linearly independent vector* $\mathbf{h}_2$ *one has*

$$\mathbf{f}(\mathbf{x} + 2\pi\mathbf{n}) = \mathbf{f}(\mathbf{x}) + n_1\mathbf{h}_1 + n_2\mathbf{h}_2, \ \forall \mathbf{x} \in \mathbb{R}^2,$$

*for all* $\mathbf{n} \in \mathbb{Z}^2$.

(vi) *If* (2.19) *is fulfilled, then* $\mathbf{f} \in \mathbf{C}^{1+\alpha}(\mathbb{R}^2)$ *and the Jacobian satisfies the estimate* $J_{\mathbf{f}}(\mathbf{x}) \ge c$ *for all* $\mathbf{x} \in \mathbb{R}^2$;

Note that the properties stated in Theorem 4.1(iv) are certainly standard for smooth, or even Lipshitz maps $\mathbf{f}$. For homeomorphisms $\mathbf{f}$ of class $\mathbf{H}^1_{\mathrm{loc}}$ such "standard" results as the chain rule or the change of variables under the integral, are not obvious. They require in addition that the functions $\mathbf{f}$ and $\mathbf{f}^{-1}$ should map sets of measure zero into sets of measure zero. This property follows from Theorem 4.1(iii) according to [13], Ch.5, §3.

Let us now state the appropriate result for an admissible domain $\Omega$ satisfying Condition 2.8. Recall that the boundary $\partial\Omega$ consists of two disjoint Jordan curves $\ell_+$ and $\ell_-$. Without loss of generality we assume that $\mathbf{0} \in \ell_-$. As defined in Sect. 2, $Z \subset \partial\Omega$ is the set of the boundary points where the smoothness of $\ell_+$, $\ell_-$ breaks down.

**Theorem 4.2.** *Suppose that an admissible domain* $\Omega$ *satisfies Condition 2.8. Let* $\mathbf{G}$ *satisfy* (2.12), (4.1) *and* (2.19). *Then there exists a unique homeomorphism* $\mathbf{f} = \mathbf{f}_\Omega = (f_1, f_2) : \Omega \to \mathcal{S} = \mathcal{S}_1$, $\mathbf{f} \in \mathbf{H}^1_{\mathrm{loc}}(\Omega)$ *such that*

(i) $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $f_1(\mathbf{x}) \to +\infty$ *as* $x_1 \to +\infty$, *and* $f_1(\mathbf{x}) \to -\infty$ *as* $x_1 \to -\infty$;

(ii) *The components* $f_1, f_2$ *satisfy the equation* (4.2) *for a.a.* $\mathbf{x} \in \Omega$.

*The map* $\mathbf{f}$ *satisfies the following properties:*

(iii) *For some number $h > 0$ one has*

$$\mathbf{f}(\mathbf{x} + 2\pi n \mathbf{e}_1) = \mathbf{f}(\mathbf{x}) + 2\pi n h \mathbf{e}_1, \ \forall \mathbf{x} \in \Omega,$$

*for all $n \in \mathbb{Z}$;*

(iv) $\mathbf{f} \in \mathbf{C}^{1+\alpha}\big(\overline{\Omega} \setminus Z\big)$, $\mathbf{f}^{-1} \in \mathbf{C}^{1+\alpha}\big(\overline{\mathbb{S}} \setminus \mathbf{f}(Z)\big)$, *and the Jacobian $J_{\mathbf{f}}(\mathbf{x})$ is positive everywhere in $\Omega$. Moreover, for each $\mathbf{x}_0 \in Z$ there exist a number $\nu \in (0, 2]$ and four non-degenerate Hölder-continuous matrix-functions $\mathbf{M}, \mathbf{T}$ and $\mathbf{\Phi}$, $\mathbf{\Psi}$ with real-valued entries such that in the vicinity of $\mathbf{x}_0$ and $\mathbf{z}_0 = \mathbf{f}(\mathbf{x}_0) \in \mathbf{f}(Z)$ one has the representations*

$$\begin{aligned} \mathfrak{D}\mathbf{f}(\mathbf{x}) &= |\mathbf{x} - \mathbf{x}_0|^{\frac{1}{\nu}-1} \mathbf{\Phi}\left(\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right)\mathbf{M}(\mathbf{x}), \\ \mathfrak{D}\mathbf{f}^{-1}(\mathbf{z}) &= |\mathbf{z} - \mathbf{z}_0|^{\nu-1} \mathbf{\Psi}\left(\frac{\mathbf{z} - \mathbf{z}_0}{|\mathbf{z} - \mathbf{z}_0|}\right)\mathbf{T}(\mathbf{z}). \end{aligned} \tag{4.3}$$

The next Lemma establishes some further properties of $\mathbf{f}$ that follow from Theorems 4.1 and 4.2.

**Lemma 4.3.** *Let the matrix $\mathbf{G}$ and the map $\mathbf{f}$ be as in Theorem 4.1 or 4.2. Then the following identities hold:*

$$\nabla f_1 = -\mathbf{J}\mathbf{G}\nabla f_2, \tag{4.4}$$

$$\langle \mathbf{F}\nabla f_1, \mathbf{F}\nabla f_2 \rangle = 0, \tag{4.5}$$

$$J_{\mathbf{f}} = |\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2; \tag{4.6}$$

$$J_{\mathbf{f}}^{-1} \mathfrak{D}\mathbf{f} \ \mathbf{G} \ \mathfrak{D}\mathbf{f}^T = \mathbf{I}. \tag{4.7}$$

*Proof.* As $\det \mathbf{F} = \det \mathbf{G} = 1$, a direct calculation shows that

$$\mathbf{J} = \mathbf{G}\mathbf{J}\mathbf{G}, \quad \mathbf{J} = \mathbf{F}\mathbf{J}\mathbf{F}.$$

Noticing also that $\mathbf{J}^2 = -\mathbf{I}$, from (4.2) we obtain (4.4) and the relation

$$\mathbf{F}\nabla f_2 = \mathbf{J}\mathbf{F}\nabla f_1.$$

This implies the orthogonality (4.5) in view of the obvious equality $\langle \mathbf{J}\boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0$, $\forall \boldsymbol{\xi} \in \mathbb{R}^2$. It also yields the equality $|\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2$. The equality (4.7) is a direct consequence of (4.5) and (4.6).

To prove (4.6) compute the Jacobian, using (4.4):

$$\begin{aligned} J_{\mathbf{f}}(\mathbf{x}) = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} &= -\langle \nabla f_1, \mathbf{J}\nabla f_2 \rangle \\ &= -\langle \nabla f_1, \mathbf{G}\mathbf{J}\mathbf{G}\nabla f_2 \rangle = \langle \nabla f_1, \mathbf{G}\nabla f_1 \rangle \\ &= |\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2. \end{aligned}$$

□

*Remark* 4.4. By (4.6) the norm $|\mathfrak{D}\mathbf{f}|$ can be estimated as follows: $|\mathfrak{D}\mathbf{f}|^2 \le K J_{\mathbf{f}}$ with some positive constant $K$ depending on the matrix $\mathbf{G}$. Recall that this inequality serves as a definition of the so-called $K$-quasi-conformal maps (see [12], Sect 12.1).

Similarly, $|(\mathfrak{D}\mathbf{f})^{-1}|^2 = \big(|\mathfrak{D}\mathbf{f}|\mathbf{J}_{\mathbf{f}}^{-1}\big)^2 \le K J_{\mathbf{f}}^{-1}$.

4.2. **Unitary transformation.** Notice that the mapping $\mathbf{f} = \mathbf{f}_{\mathbb{R}^2}$ constructed in Theorem 4.1 transforms the lattice $\mathbf{\Gamma} = (2\pi\mathbb{Z})^2$ into the lattice generated by the vectors $\mathbf{h}_1, \mathbf{h}_2$. It is slightly more convenient to reduce this lattice back to $\mathbf{\Gamma}$ by applying the non-degenerate linear transformation $\mathbf{R} = \mathbf{R}_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the following two relations:

$$\mathbf{R}\mathbf{h}_1 = 2\pi\mathbf{e}_1,\ \mathbf{R}\mathbf{h}_2 = 2\pi\mathbf{e}_2.$$

By Theorem 4.1(v) the composite mapping $\mathbf{g} = \mathbf{R}\circ\mathbf{f}$ satisfies

$$\mathbf{g}(\mathbf{x} + 2\pi\mathbf{n}) = \mathbf{g}(\mathbf{x}) + 2\pi n_1\mathbf{e}_1 + 2\pi n_2\mathbf{e}_2, \quad \forall \mathbf{x}\in\mathbb{R}^2,\ \forall\mathbf{n}\in\mathbb{Z}^2. \tag{4.8}$$

Note that in view of (4.7), we have

$$\mathbf{A} := J_{\mathbf{g}}^{-1}\mathfrak{D}\mathbf{g}\ \mathbf{G}\ \mathfrak{D}\mathbf{g}^T = (\det\mathbf{R})^{-1}\mathbf{R}\mathbf{R}^T. \tag{4.9}$$

Similarly, in Theorem 4.2 the one-dimensional lattice $\boldsymbol{\gamma}_1$ is transformed into the lattice $\boldsymbol{\gamma}_1^h = \{2\pi n h\mathbf{e}_1\}, n\in\mathbb{Z}$. We rescale $\boldsymbol{\gamma}_1^h$ back to $\boldsymbol{\gamma}_1$ by applying the transformation $\mathbf{R} = \mathbf{R}_\Omega : \mathcal{S}\to\mathcal{S}$ defined as follows:

$$\mathbf{R}\mathbf{e}_2 = \mathbf{e}_2,\ \mathbf{R}\mathbf{e}_1 = h^{-1}\mathbf{e}_1,$$

Then clearly, the mapping $\mathbf{g} = \mathbf{R}\circ\mathbf{f}$ satisfies the relation

$$\mathbf{g}(\mathbf{x} + 2\pi n\mathbf{e}_1) = \mathbf{g}(\mathbf{x}) + 2\pi n\mathbf{e}_1,\ \forall\mathbf{x}\in\Omega,\quad \forall n\in\mathbb{Z}. \tag{4.10}$$

Note also that the matrix $\mathbf{A}$ in (4.9) is diagonal in this case. Depending on the context, below we use either the notation $\mathbf{g}, \mathbf{A}$ or $\mathbf{g}_\Lambda, \mathbf{A}_\Lambda$ where $\Lambda$ assumes one of the two values: $\mathbb{R}^2$ or $\Omega$.

Denote

$$\tilde\Lambda = \begin{cases}\Lambda, & \text{if}\ \ \Lambda = \mathbb{R}^2\ \ \text{or}\ \ \mathcal{C};\\ \mathcal{S}, & \text{if}\ \ \Lambda = \Omega.\end{cases}$$

In the case $\Lambda = \Omega$, under the conditions of Theorem 4.2 one can easily show that the set $\tilde{\mathbf{\Sigma}} = \mathbf{g}(\mathbf{\Sigma})$ is again a system of curves. More precisely, if $\ell_j, j = 1, 2, \dots, N$ are the $\mathsf{C}^1$-arcs from Definition 2.2 (see also end of Sect. 2), then each $\mathbf{g}(\ell_j)$ is again a $\mathsf{C}^1$-arc satisfying all the required properties. In the case of curves $\ell_j$ ending or starting at the points of $Z$ this is done by a suitable re-parametrisation, using (4.3). Recall that $\mathbf{g}(\mathbf{\Sigma})$ contains the boundary of the strip $\mathcal{S}$.

Introduce also the sets $\tilde\Upsilon$, $\tilde{\mathbf{\Xi}}$ defined similarly to (2.4) and (2.6) with $\tilde\Lambda$ and $\tilde{\mathbf{\Sigma}}$ instead of $\Lambda$ and $\mathbf{\Sigma}$. Note that according to Theorem 4.2 the following is fulfilled for the mappings $\mathbf{g}, \mathbf{g}^{-1}$ in the case $\Lambda = \Omega$:

$$\mathbf{g}\in\ \mathsf{W}^{1,\tau}(\Omega'),\ \mathbf{g}^{-1}\in\mathsf{W}^{1,\tau}(\Omega''),\quad \mathfrak{D}\mathbf{g}^{-1}\big|_{\tilde{\mathbf{\Xi}}}\in\mathsf{L}^{\tau/2}(\tilde{\mathbf{\Xi}}) \tag{4.11}$$

for any bounded domains $\Omega'\subset\Omega$, $\Omega''\subset\mathcal{S}$ and some $\tau > 2$.

Using Theorem 4.1(iii), (iv) and Theorem 4.2 (iv), it is easy to show that the operator

$$(Su)(\mathbf{x}) = u\big(\mathbf{g}^{-1}(\mathbf{x})\big),\ u \in \mathsf{L}^2(\Lambda),$$

is unitary from $\mathsf{L}^2(\Lambda)$ onto the space $\mathsf{L}^2(\tilde{\mu}, \tilde{\Lambda})$ with the weight

$$\tilde{\mu}(\mathbf{x}) = \Big(J_{\mathbf{g}}\big(\mathbf{g}^{-1}(\mathbf{x})\big)\Big)^{-1} = J_{\mathbf{g}^{-1}}(\mathbf{x}).$$

Denote

$$\begin{cases} \tilde{\mathbf{a}}(\mathbf{x}) = \big(((\mathfrak{D}\mathbf{g}^T)^{-1}\mathbf{a}) \circ \mathbf{g}^{-1}\big)(\mathbf{x}), \\ \tilde{V}(\mathbf{x}) = \big((J_{\mathbf{g}}^{-1} V) \circ \mathbf{g}^{-1}\big)(\mathbf{x}), \end{cases} \qquad \mathbf{x} \in \tilde{\Lambda},$$

and in the case $\Lambda = \Omega$ denote

$$\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \dots \tilde{\sigma}_N),$$

$$\tilde{\sigma}_j(\mathbf{x}) = |(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})|(\sigma_j \circ \mathbf{g}^{-1})(\mathbf{x}), \quad \mathbf{x} \in \tilde{\Sigma}_j,\ j = 1, 2, \dots, N,$$

where by $\mathbf{t}_j(\mathbf{x})$ we have denoted the unit tangent vector to $\tilde{\Sigma}_j$ at the point $\mathbf{x} \in \tilde{\Sigma}_j$.

In the next Theorem, among other properties of the map $S$ we show that the coefficients $\tilde{V}, \tilde{\mu}$, $\tilde{\mathbf{a}}$ and $\tilde{\boldsymbol{\sigma}}$ satisfy the conditions of Theorems 2.6 or 2.9.

**Theorem 4.5.** *Let $S$ be as defined above. Then*

(i) *Under the conditions of Theorems 2.6 or 2.9 one has $\tilde{\mu} \in \mathsf{L}^{\tau/2}(\tilde{\Upsilon})$,*

$$\tilde{V} \in \mathsf{L}^{\tilde{p}}(\tilde{\Upsilon}), \quad \tilde{p} = \frac{p\tau}{2(p-1)+\tau},$$
$$\tilde{\mathbf{a}} \in \mathbf{L}^{\tilde{s}}(\tilde{\Upsilon}), \quad \tilde{s} = \frac{s\tau}{s-2+\tau},$$
$$\tilde{\boldsymbol{\sigma}} \in \mathbf{L}^{\tilde{r}}(\tilde{\boldsymbol{\Xi}}), \quad \tilde{r} = \frac{r\tau}{2(r-1)+\tau}.$$

*where $\tau > 2$ is as in Theorem 4.1(*iii*) or* (4.11)*. The exponents $\tilde{p}$ and $\tilde{s}$ satisfy the inequalities $\tilde{p} > 1$, $\tilde{s} > 2$, $\tilde{r} > 1$;*

(ii) *The map $S$ (resp. $S^{-1}$) is bounded as an operator from $\mathsf{H}^1(\Lambda)$ to $\mathsf{H}^1(\tilde{\Lambda})$ (resp. $\mathsf{H}^1(\tilde{\Lambda})$ to $\mathsf{H}^1(\Lambda)$), and from $\mathsf{H}^1_0(\Omega)$ to $\mathsf{H}^1_0(\mathbb{S})$ (resp. $\mathsf{H}^1_0(\mathbb{S})$ to $\mathsf{H}^1_0(\Omega)$).*

(iii) *Let $\mathbf{A}$ be as defined in* (4.9)*. Then the following unitary equivalence holds:*

$$SH_{\aleph}(\mathbf{G}, \mathbf{a}, V, 1, \boldsymbol{\sigma}; \Lambda)S^* = H_{\aleph}(\mathbf{A}, \tilde{\mathbf{a}}, \tilde{V}, \tilde{\mu}, \tilde{\boldsymbol{\sigma}}; \tilde{\Lambda}).$$

*Proof.* (i) The inequalities $\tilde{p} > 1$, $\tilde{s} > 2$ and $\tilde{r} > 1$ immediately result from the conditions $p > 1,\ s > 2, r > 1,\ \tau > 2$ by inspection.

It follows from Theorem 4.1(iii) and (v) or (4.11), that $\tilde{\mu} \in \mathsf{L}^{\tau/2}(\tilde{\Upsilon})$. Let us prove that $\tilde{V} \in L^{\tilde{p}}(\tilde{\Upsilon})$. By Hölder's inequality for any bounded domain $\Omega_1 \subset \mathbb{R}^2$ we have

$$\int_{\Omega_1} |J_{\mathbf{g}}^{-1} V \circ \mathbf{g}^{-1}(\mathbf{x})|^{\tilde{p}} d\mathbf{x} \\ \leq \left[\int_{\Omega_1} |V \circ \mathbf{g}^{-1}(\mathbf{x})|^{\tilde{p}\beta^{-1}} J_{\mathbf{g}^{-1}}(\mathbf{x}) d\mathbf{x}\right]^{\beta} \left[\int_{\Omega_1} \Big(J_{\mathbf{g}^{-1}}(\mathbf{x})\Big)^{(\tilde{p}-\beta)(1-\beta)^{-1}} d\mathbf{x}\right]^{1-\beta}$$

with $\beta = \tilde{p}p^{-1}$. Noticing that $(\tilde{p} - \beta)(1 - \beta)^{-1} = \tau/2$, and using Theorem 4.1(iii), (iv) or (4.11), we conclude that the r.h.s. of the last inequality does not exceed

$$C\left[\int_{\mathbf{g}^{-1}(\Omega_1)} |V(\mathbf{x})|^p d\mathbf{x}\right]^{\beta} \left[\int_{\Omega_1} \Big(J_{\mathbf{g}^{-1}}(\mathbf{x})\Big)^{\tau/2} d\mathbf{x}\right]^{1-\beta} < \infty.$$

To prove that $\mathbf{a} \in \mathbf{L}^{\tilde{s}}(\tilde{\Upsilon})$ note that in view of Remark 4.4,

$$\int_{\Omega_1} |\tilde{\mathbf{a}}(\mathbf{x})|^{\tilde{s}} d\mathbf{x} \leq C \int_{\Omega_1} \left[J_{\mathbf{g}^{-1}}(\mathbf{x}) \big|\big(\mathbf{a} \circ \mathbf{g}^{-1}\big)(\mathbf{x})\big|^2\right]^{\tilde{s}/2} d\mathbf{x}.$$

Now, repeating the argument used in the first part of the proof, we arrive at the required property.

A similar calculation can be done for the function $\boldsymbol{\sigma}$. Precisely, for any $\mathsf{C}^1$-arc $\ell \subset \tilde{\Sigma}_j$ we have

$$\int_{\ell} \Big[|(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})| \ \ |(\sigma_j \circ \mathbf{g}^{-1})(\mathbf{x})|\Big]^{\tilde{r}} dS \\ \leq \left[\int_{\ell} \Big(|(\sigma_j \circ \mathbf{g}^{-1})(\mathbf{x})|\Big)^{\tilde{r}\gamma^{-1}} |(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})| dS\right]^{\gamma} \\ \left[\int_{\ell} \Big(|(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})|\Big)^{(\tilde{r}-\gamma)(1-\gamma)^{-1}} dS\right]^{1-\gamma}$$

with $\gamma = \tilde{r}r^{-1}$. Noticing that $(\tilde{r} - \gamma)(1 - \gamma)^{-1} = \tau/2$, we conclude that the r.h.s. of the last inequality does not exceed

$$C\left[\int_{\mathbf{g}^{-1}(\ell)} |\sigma_j(\mathbf{x})|^r dS\right]^{\gamma} \left[\int_{\ell} |\mathfrak{D}\mathbf{g}^{-1}(\mathbf{x})|^{\tau/2} dS\right]^{1-\gamma},$$

and therefore, by (4.11) it is bounded.

(ii) Let $u \in \mathsf{H}^1(\Lambda)$ and $v = Su$. Let us first prove that $\nabla v \in \mathsf{L}^2(\tilde\Lambda)$. Since the matrix $\mathbf{A}$ defined in (4.9) is positive-definite, changing the variables, we have

$$c\|\nabla v\|^2 \le \int_{\tilde\Lambda} \langle \mathbf{A}\nabla v, \overline{\nabla v}\rangle d\mathbf{x} = \int_{\tilde\Lambda} \langle \mathbf{G}\nabla u, \overline{\nabla u}\rangle \circ \mathbf{g}^{-1} J_{\mathbf{g}^{-1}} d\mathbf{x}$$

$$= \int_{\Lambda} \langle \mathbf{G}\nabla u, \overline{\nabla u}\rangle d\mathbf{x} \le C\|\nabla u\|^2.$$

In the last inequality we have used (2.12), and to secure the change of variables in the case $\Lambda = \mathbb{R}^2$ we refer to Theorem 4.1(iv).

Since $\tilde\mu = J_{\mathbf{g}^{-1}} \in \mathsf{L}^{\tau/2}(\tilde\Upsilon)$ with $\tau > 2$, in order to prove the boundedness of the operator $S : \mathsf{H}^1(\Lambda) \to \mathsf{H}^1(\tilde\Lambda)$ it remains to use the unitarity of $S : \mathsf{L}^2(\Lambda) \to \mathsf{L}^2(\tilde\mu, \tilde\Lambda)$ and Proposition 2.4. Similarly for $S^{-1}$.

To show that $S$ maps $\mathsf{H}^1_0(\Omega)$ into $\mathsf{H}^1_0(\mathcal{S})$, it suffices to notice that for any $u \in \mathsf{C}^1_0(\Omega)$ we have $u \circ \mathbf{g}^{-1} \in \mathsf{C}^1_0(\mathcal{S})$.

(iii) The result follows by a straightforward calculation. □

*Proof of Theorem 2.6.* As it was explained in Remark 2.7, Theorem 2.6 was established in [7] for constant matrices $\mathbf{G}$ and $\omega = \mu = 1$. In view of the unitary equivalence established in Theorem 4.5(iii), and by Corollary 3.2, Theorem 2.6 for general $\mathbf{G}$ and $\omega, \mu$ follows immediately from [7]. □

As far as Theorem 2.9 is concerned, by Theorem 4.5(iii) and Corollary 3.2, it will result from

**Theorem 4.6.** *Suppose that $\Omega = \mathcal{S}_1$, $\omega = \mu = 1$. Let the conditions* (2.10), (2.11) *be satisfied, and let $\mathbf{G} = \mathbf{B}$ be a constant diagonal matrix. Then the spectra of the operators $H_D(\mathbf{B}, \mathbf{a}, V, \boldsymbol{\sigma}; \Omega)$, $H_N(\mathbf{B}, \mathbf{a}, V, \boldsymbol{\sigma}; \Omega)$ are absolutely continuous.*

## 5. Proof of Theorems 4.6 and 2.9

From now on we assume that the conditions of Theorem 4.6 are fulfilled.

5.1. **Extension operators.** The proof is based on a reduction of $H_D$ and $H_N$ to a periodic operator. To construct the appropriate operator we reflect the strip $\mathcal{S}_1$ in the line $x_2 = 0$ and extend the functions $\mathbf{a}, V$ and $\sigma$ into the lower half of the obtained domain. Precisely, denote

$$\mathcal{S}_u = \mathcal{S}_1, \quad \mathcal{S}_l = \{\mathbf{x} \in \mathbb{R}^2 : (x_1, -x_2) \in \mathcal{S}_u\},$$
$$\mathcal{S}_0 = \mathcal{S}_u \cup \mathcal{S}_l \cup \{\mathbf{x} : x_2 = 0\},$$

and define subspaces

$$\mathsf{L}^2_{\pm}(\mathcal{C}) = \{u \in \mathsf{L}^2(\mathcal{C}) : u(x_1, x_2) = \pm u(x_1, -x_2), \quad \text{a.a. } \mathbf{x} \in \mathcal{S}_0\}$$

of all even ($\mathsf{L}^2_+$) and all odd ($\mathsf{L}^2_-$) functions from $\mathsf{L}^2(\mathcal{C})$. One easily concludes by inspection that the projections onto these subspaces are given by the formula

$$\mathcal{P}_\pm u = \frac{1}{2}\big(u(x_1, x_2) \pm u(x_1, -x_2)\big). \tag{5.1}$$

We shall also need *the extension operators* $W_\pm : \mathsf{L}^2(\mathcal{S}_u) \to \mathsf{L}^2_\pm(\mathcal{C})$. For $\mathbf{x} \in \mathcal{S}_0$ they are defined as follows:

$$(W_\pm u)(x_1, x_2) = \begin{cases} u(x_1, x_2)/\sqrt{2}, \ \mathbf{x} \in \mathcal{S}_u; \\ \pm u(x_1, -x_2)/\sqrt{2}, \ \mathbf{x} \in \mathcal{S}_l, \end{cases}$$

and extended to $\mathbb{R}^2$ as $\boldsymbol{\gamma}_2$-periodic functions. Using the formula

$$(W_\pm^{-1} u)(\mathbf{x}) = \sqrt{2}\ u(\mathbf{x}), \mathbf{x} \in \mathcal{S}_u, \quad \forall u \in \mathsf{L}^2_\pm(\mathcal{C}), \tag{5.2}$$

one readily proves that $W_\pm$ is unitary on $\mathsf{L}^2(\mathcal{S}_u)$. To find out how the functions from the Sobolev spaces $\mathsf{H}^1(\mathcal{S}_u)$, $\mathsf{H}^1_0(\mathcal{S}_u)$ transform under the extensions $W_\pm$, denote

$$\mathsf{H}^1_\pm(\mathcal{C}) = \mathcal{P}_\pm \mathsf{H}^1(\mathcal{C}).$$

Given the explicit form (5.1) of the operator $\mathcal{P}_\pm$, it is easy to see that

$$\mathsf{H}^1_\pm(\mathcal{C}) = \mathsf{H}^1(\mathcal{C}) \cap \mathsf{L}^2_\pm(\mathcal{C}). \tag{5.3}$$

Now, observing that

$$u(x_1, -\pi) = u(x_1, 0) = u(x_1, \pi) = 0, \ \text{a.a. } x_1 \in \mathbb{R}, \tag{5.4}$$

for all $u \in \mathsf{H}^1_-(\mathcal{C})$, one sees that

$$W_+ \mathsf{H}^1(\mathcal{S}_u) = \mathsf{H}^1_+(\mathcal{C}), \quad W_- \mathsf{H}^1_0(\mathcal{S}_u) = \mathsf{H}^1_-(\mathcal{C}). \tag{5.5}$$

5.2. **Reduction.** Now we can describe the periodic operator associated with the operators $H_D$ and $H_N$. Assume that the conditions of Theorem 4.6 are satisfied. We begin with definition of the corresponding system of curves. Define $\boldsymbol{\Sigma}_u = (\Sigma_{1,u}, \Sigma_{2,u}, \dots, \Sigma_{N,u})$, where $\Sigma_{j,u} = \Sigma_j, j = 1, 2, \dots, N$, and

$$\boldsymbol{\Sigma}_l = (\Sigma_{1,l}, \Sigma_{2,l}, \dots, \Sigma_{N,l}),$$
$$\Sigma_{j,l} = \{\mathbf{x} \in \overline{\mathcal{S}_l} : (x_1, -x_2) \in \Sigma_{j,u}\},$$
$$\boldsymbol{\Sigma}_0 = (\boldsymbol{\Sigma}_u, \boldsymbol{\Sigma}_l).$$

By Definition 2.2 $\boldsymbol{\Sigma}_0$ is a system of curves in $\mathcal{C}$. We emphasise that this system contains two copies of the upper and lower boundaries of the strip $\mathcal{S}_1$.

Define the $\boldsymbol{\gamma}_2$-periodic functions $\mathbf{b}, Q$ by applying the extension operators $W_\pm$ to $\mathbf{a}, V$ in the following way:

$$Q = \sqrt{2}\ W_+ V, \quad b_1 = \sqrt{2}\ W_+ a_1, \quad b_2 = \sqrt{2}\ W_- a_2.$$

It is clear that the functions $Q, b_1$ are even, and $b_2$ is odd. Clearly, the new coefficients $\mathbf{b}, Q$ satisfy (2.10) with $\Upsilon = \mathbb{T}^2$. In a similar way we extend to $\boldsymbol{\Sigma}_0$ the function $\boldsymbol{\sigma}$, i.e. let $\boldsymbol{\rho}$ be the function on $\boldsymbol{\Sigma}_0$ defined as follows:

$$\boldsymbol{\rho}(\mathbf{x}) = \begin{cases} \boldsymbol{\sigma}(\mathbf{x}), & \mathbf{x} \in \boldsymbol{\Sigma}_u, \\ \boldsymbol{\sigma}(x_1, -x_2), & \mathbf{x} \in \boldsymbol{\Sigma}_l, \end{cases}$$

and extended $\boldsymbol{\gamma}_2$-periodically. Clearly, $\boldsymbol{\rho}$ satisfies (2.11) with $\boldsymbol{\Xi} = \boldsymbol{\Sigma}_0/\boldsymbol{\gamma}_1$.

Now define the reference periodic operator $H = H_P(1, \mathbf{B}, \mathbf{b}, Q, 1, \boldsymbol{\rho}; \mathcal{C})$. Using the symmetry properties of $\mathbf{b}, Q, \boldsymbol{\rho}$, we decompose the operator $H$ in the orthogonal sum associated with the subspaces $\mathsf{L}^2_\pm = \mathsf{L}^2_\pm(\mathcal{C})$.

**Lemma 5.1.** *The subspaces $\mathsf{L}^2_\pm$ are invariant subspaces of the operator $H_P$.*

*Proof.* We need to check that

$$\mathcal{P}_\pm D[h] = \mathsf{L}^2_\pm \cap D[h], \tag{5.6}$$

and that for any $u, v \in D[h]$

$$h[u, v] = h[\mathcal{P}_+ u, \mathcal{P}_+ v] + h[\mathcal{P}_- u, \mathcal{P}_- v]. \tag{5.7}$$

The equality (5.6) follows from (5.3). To prove (5.7) it will suffice to verify that

$$h[\mathcal{P}_+ u, \mathcal{P}_- v] = 0, \ \forall u, v \in \mathsf{H}^1(\mathcal{C}). \tag{5.8}$$

Write out the l.h.s. using the notation $w_\pm = \mathcal{P}_\pm w$:

$$\sum_{l=1}^{2} \big(b_{ll}(D_l - b_l)u_+, (D_l - b_l)v_-\big) + (Qu_+, v_-) + \int_{\boldsymbol{\Sigma}_0} \boldsymbol{\rho} u_+ \overline{v_-} dS.$$

Two last terms vanish since $Q$ and $\boldsymbol{\rho}$ are even functions. In particular, the integrals over $\partial\mathcal{S}$ equal zero in view of (5.4). To handle the first term make the following table using the properties of the coefficients:

$$\begin{array}{llll} (D_1 - b_1)u_+ & \text{even,} & (D_1 - b_1)v_- & \text{odd,} \\ (D_2 - b_2)u_+ & \text{odd,} & (D_2 - b_2)v_- & \text{even.} \end{array}$$

Now it is easy to see that the first term also vanishes, which implies (5.8). □

Denote the parts of the operator $H$ in this orthogonal decomposition by $H_+$ and $H_-$. It can be shown that $H_\pm$ are unique self-adjoint operators associated with the closed quadratic forms $h_\pm[\,\cdot\,]$ obtained from $h[\,\cdot\,]$ by restricting the domain $D[h]$ to $D[h_\pm] = \mathsf{H}^1_\pm(\mathcal{C})$. The final step of the reduction of $H_D$ and $H_N$ to $H$ is implemented in the following Lemma:

**Lemma 5.2.** *Let the conditions of Theorem 4.6 be satisfied. Then*

$$W_- H_D W_-^* = H_-, \quad W_+ H_N W_+^* = H_+. \tag{5.9}$$

*Proof.* By (5.5) $W_+\mathcal{D}_N = D[h_+]$ and $W_-\mathcal{D}_D = D[h_-]$, and hence it suffices to show that the bilinear forms of the operators in (5.9), coincide on the domains $D[h_-] = \mathsf{H}^1_-(\mathcal{C})$ and $D[h_+] = \mathsf{H}^1_+(\mathcal{C})$ respectively. Let $u_\pm, v_\pm \in \mathsf{H}^1_\pm(\mathcal{C})$. Using (5.2) and referring again to the symmetry properties of the coefficients, one can write:

$$\begin{aligned}
&\big(\mathbf{B}(\mathbf{D}-\mathbf{a})W_\pm^* u_\pm, (\mathbf{D}-\mathbf{a})W_\pm^* v_\pm\big) \\
&\qquad + \big(VW_\pm^* u_\pm, W_\pm^* v_\pm\big) + \int_{\mathbf{\Sigma}} \boldsymbol{\sigma} W_\pm^* u_\pm \overline{W_\pm^* v_\pm} dS \\
&= 2\int_{\mathcal{S}_u} \Big[\sum_{l=1}^2 b_{ll}(D_l - a_l)u_\pm \overline{(D_l - a_l)v_\pm} + V u_\pm \overline{v_\pm}\Big] d\mathbf{x} + 2\int_{\mathbf{\Sigma}} \boldsymbol{\sigma} u_\pm \overline{v_\pm} dS \\
&= \int_{\mathcal{S}_0} \Big[\sum_{l=1}^2 b_{ll}(D_l - b_l)u_\pm \overline{(D_l - b_l)v_\pm} + Q u_\pm \overline{v_\pm}\Big] d\mathbf{x} + \int_{\mathbf{\Sigma}_0} \boldsymbol{\rho} u_\pm \overline{v_\pm} dS.
\end{aligned}$$

This form coincides with the bilinear form of the operator $H_\pm$. □

*Proof of Theorem 4.6.* The coefficients $\mathbf{b}, Q$ satisfy the conditions of Proposition 3.3. Therefore, the periodic operator $H$ is absolutely continuous, and so are the orthogonal parts $H_+$ and $H_-$. By virtue of Lemma 5.2 the operators $H_N$ and $H_D$ are unitarily equivalent to $H_+, H_-$ and thus they are also absolutely continuous, as required. □

Theorem 4.6 combined with Corollary 3.2 and Theorem 4.5 leads to Theorem 2.9.

## 6. Quasi-conformal maps

In this section we prove Theorems 4.1 and 4.2. We are using the standard approach to second order elliptic equations in dimension two, which consists in passing to the complex variable and using the theory of quasi-analytic functions (see e.g. [3], [35]). Let us define $z = x_1 + ix_2$, $f = f_1 + if_2$.

To begin with we notice that the equation (4.2) for $f_1$, $f_2$ is equivalent to the *Beltrami* equation (see [35]):

$$\partial_{\bar z} f = q\partial_z f, \tag{6.1}$$

where

$$\partial_z = \frac{1}{2}\big(\partial_{x_1} - i\partial_{x_2}\big),\ \partial_{\bar z} = \frac{1}{2}\big(\partial_{x_1} + i\partial_{x_2}\big)$$

with the complex-valued function

$$q = \frac{-g_{12} + i(1 - g_{22})}{g_{12} - i(g_{22} + 1)} \tag{6.2}$$

Note that $\|q\|_{\mathsf{L}^\infty}$ is strictly less than 1, and that the Jacobian of $\mathbf{f}$ satisfies the relation:

$$J_{\mathbf{f}} = |\partial_z f|^2 - |\partial_{\bar z} f|^2 = (1 - |q|^2)|\partial_z f|^2.$$

We say that a continuous function $f$ is $q$-quasi-conformal (or quasi-conformal) if $\partial_z f \in \mathsf{L}^2_{\mathrm{loc}}$ and $f$ satisfies the equation (6.1) for a.a. $z$.

6.1. **Mappings of $\mathbb{R}^2$.** The following theorem is a one-to-one "translation" of Theorem 4.1 into the language of quasi-conformal mappings, except for additional item (v).

**Theorem 6.1.** *Let $q \in \mathsf{L}^\infty(\mathbb{C})$ be a function such that $\|q\|_{\mathsf{L}^\infty} < 1$. Suppose that $q$ is periodic, that is,*

$$q(z) = q(z + 2\pi) = q(z + 2\pi i), \text{ a. a. } z \in \mathbb{C}.$$

*Then there exists a unique $q$-quasi-conformal homeomorphism $f$ of the complex plane onto itself such that $f(0) = 0$, $f(2\pi) = 2\pi$, and $f(\infty) = \infty$.*

*Moreover,*

(i) *$\partial_z f \neq 0$ almost everywhere;*
(ii) *There exists a number $\tau > 2$ such that $f, f^{-1} \in \mathsf{W}^{1,\tau}_{\rm loc}(\mathbb{C})$;*
(iii) *For any $u \in \mathsf{H}^1_{\rm loc}(\mathbb{C})$ the derivatives $\partial_z(u \circ f)$ and $\partial_{\bar z}(u \circ f)$ are found by the standard chain rule. Also, for any $h \in \mathsf{L}^1_{\rm loc}(\mathbb{C})$ the function $h \circ \mathbf{f}$ belongs to $\mathsf{L}^1_{\rm loc}(J_{\mathbf{f}}, \mathbb{C})$ and*

$$\int_{f(\Omega_1)} h(\mathbf{x}) d\mathbf{x} = \int_{\Omega_1} \big(h \circ f\big)(\mathbf{x}) J_{\mathbf{f}}(\mathbf{x}) d\mathbf{x}$$

*for any open bounded $\Omega_1 \subset \mathbb{C}$;*
(iv) *The mapping $f$ has the following periodicity property:*

$$f(z + 2\pi n + 2i\pi m) = f(z) + 2\pi n + \varkappa m, \ \forall z \in \mathbb{C}, \quad \forall m, n, \in \mathbb{Z}, \tag{6.3}$$

*with some $\varkappa$ which has a non-zero imaginary part;*
(v) *If $q(\bar z) = \overline{q(z)}$ almost everywhere, then $\varkappa$ in (6.3) is purely imaginary: $\varkappa = 2i \, \mathrm{Im} f(\pi i)$.*
(vi) *If $q \in \mathsf{C}^\alpha(\mathbb{C})$, $0 < \alpha < 1$, then $f \in \mathsf{C}^{1+\alpha}(\mathbb{C})$ and $|\partial_z f| > 0$ for all $z \in \mathbb{C}$.*

*Proof.* By Proposition 7.1 the mapping with the fixed three points $0, 2\pi, \infty$ exists, is unique, and satisfies properties (i), (ii), (iii) and (vi). We need only to prove (6.3) and (v). Let $\zeta$ be either $2\pi$ or $2\pi i$. Along with $f$ the function

$$\tilde f(z) = 2\pi \frac{f(z + \zeta) - f(\zeta)}{f(2\pi + \zeta) - f(\zeta)}$$

is also a solution of the equation (6.1), due to the periodicity of $q$. Note that the denominator here is not zero as the function $f$ is a homeomorphism, and for the same reason $f(\zeta) \neq 0$ as well. Notice also that $\tilde f(0) = 0$, $\tilde f(2\pi) = 2\pi$ and $\tilde f(\infty) = \infty$. By uniqueness of such a solution, $\tilde f = f$, or, which is the same,

$$f(z + \zeta) = c_1 f(z) + c_2$$

with

$$c_1 = \frac{f(2\pi + \zeta) - f(\zeta)}{2\pi}, \quad c_2 = f(\zeta).$$

Let us consider separately four possibilities: $|c_1| < 1$, $|c_1| > 1$, $c_1 = e^{i\theta}, \theta \in (0, 2\pi)$ and $c_1 = 1$. We shall eliminate the first three of them, thus proving that $c_1 = 1$.

Case 1: $|c_1| < 1$. For any integer $n \geq 1$ we have

$$f(n\zeta) = c_2 \sum_{k=0}^{n-1} c_1^k = c_2 \frac{1 - c_1^n}{1 - c_1},$$

and hence $f(n\zeta) \to c_2(1 - c_1)^{-1}$ as $n \to \infty$. This contradicts the fact that $f(\infty) = \infty$.

Case 2: $|c_1| > 1$. The sought contradiction follows from Case 1 by rewriting

$$f(z - \zeta) = \frac{1}{c_1} f(z) - \frac{c_2}{c_1},$$

and noting that $|c_1^{-1}| < 1$.

Case 3: $c_1 = e^{i\theta}$, $\theta \in (0, 2\pi)$. Again, as in Case 1, we have

$$f(n\zeta) = c_2 \frac{1 - c_1^n}{1 - c_1}.$$

The r.h.s. remains bounded as $n \to \infty$, which contradicts the requirement that $f(\infty) = \infty$.

Consequently, the only possible option is $c_1 = 1$.

Note that $f(2\pi) = 2\pi$ by definition of $f$, so that (6.3) with $m = 0$ is proved. Let us prove now that the imaginary part of $\varkappa = f(2\pi i)$ is non-zero. Suppose, on the contrary, that $\operatorname{Im} \varkappa = 0$. Then for any integer $m$ one can find another integer $n = n(m)$ such that $|2\pi n + \varkappa m| \leq 2\pi$, so that the r.h.s. of the equality

$$f(2\pi n + 2\pi m i) = 2\pi n + \varkappa m$$

remains bounded as $m \to \infty$. On the other hand $|2\pi n + 2\pi m i| \to \infty$ as $m \to \infty$. Again we get the same contradiction, which proves that $\operatorname{Im} \varkappa \neq 0$.

It is left to prove (v). Let $\varphi(z) := \overline{f(\overline{z})}$. Then $\varphi(0) = 0$, $\varphi(2\pi) = 2\pi$ and $\varphi(\infty) = \infty$. Further, it is easy to see (see e.g. [1], Ch. I, Sect. C, (1)) that

$$\partial_z(f(\overline{z})) = (\partial_{\overline{z}} f)(\overline{z}) = (q \partial_z f)(\overline{z}) = q(\overline{z})(\partial_z f)(\overline{z}) = q(\overline{z}) \partial_{\overline{z}}(f(\overline{z})).$$

Taking the complex conjugates of both sides and using the equality $q(\overline{z}) = \overline{q(z)}$, we obtain $\partial_{\overline{z}} \varphi(z) = q(z) \partial_z \varphi(z)$. By uniqueness we then conclude that $\varphi \equiv f$, i.e. $\overline{f(\overline{z})} = f(z)$, $\forall z \in \mathbb{C}$, and in particular, $f(-\pi i) = \overline{f(\pi i)}$. On the other hand (6.3) implies

$$f(\pi i) = f(-\pi i + 2\pi i) = f(-\pi i) + \varkappa = \overline{f(\pi i)} + \varkappa.$$

Hence $\varkappa = 2i \operatorname{Im} f(\pi i)$. □

Theorem 4.1 follows immediately.

We emphasise again that the crucial periodicity property (6.3) of the quasi-conformal map $f$ is a direct consequence of the uniqueness in Proposition 7.1. Besides, we have included in Theorem 6.1 statement (v) which also follows from the uniqueness. Using (6.2) one easily sees that the condition $q(\overline{z}) = \overline{q(z)}$ in Part (v) is equivalent to the following symmetry conditions on the matrix $\mathbf{G}$:

$$g_{jj}(x_1, x_2) = g_{jj}(x_1, -x_2), \ j = 1, 2,$$
$$g_{jl}(x_1, x_2) = -g_{jl}(x_1, -x_2), \ j \neq l.$$

Then $\operatorname{Re} \varkappa = 0$ means that the isothermal change of variables transforms the initial square lattice $(2\pi\mathbb{Z})^2$ into another *orthogonal* lattice. Although this observation is not needed in this paper, we consider it to be worth mentioning.

6.2. **Mapping of the domain $\Omega$.** The next theorem restates Theorem 4.2 in the language of quasi-conformal mappings.

**Theorem 6.2.** *Let $\Omega$ be a domain as in Theorem 4.2, in particular Condition 2.8 is fulfilled and $0 \in \ell_-$. Let $q \in \mathsf{C}^\alpha(\overline{\Omega})$ be a periodic function, i.e. $q(z) = q(z + 2\pi)$, $\forall z \in \Omega$, such that $\|q\|_{\mathsf{L}^\infty} < 1$. Then there exists a unique $q$-quasi-conformal homeomorphism $f$ of the domain $\Omega$ onto the strip $\mathcal{S}_1$ such that $f(0) = 0$, $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. The map $f$ has the following properties:*

(i) *For all $z \in \Omega$*

$$f(z + 2\pi n) = f(z) + \varkappa n, \ \forall n \in \mathbb{Z}, \tag{6.4}$$

*with some $\varkappa > 0$;*

(iii) *$f \in \mathsf{C}^{1+\alpha}(\overline{\Omega} \setminus Z)$, $f^{-1} \in \mathsf{C}^{1+\alpha}(\overline{\mathcal{S}} \setminus f(Z))$ and $|\partial_z f| > 0$ everywhere in $\Omega$. Moreover, for each $\mathbf{z}_0 \in Z$ there exist a number $\nu \in (0, 2]$ and four Hölder-continuous functions $M, T$ and $\Phi, \Psi$, separated from zero, such that in the vicinity of $z_0$ and $\zeta_0 = f(z_0) \in f(Z)$ one has the representations*

$$\partial_z f(z) = |z - z_0|^{\frac{1}{\nu} - 1} \Phi\big(\arg(z - z_0)\big) M(z),$$
$$\partial_\zeta f^{-1}(\zeta) = |\zeta - \zeta_0|^{\nu - 1} \Psi\big(\arg(\zeta - \zeta_0)\big) T(\zeta).$$

*Proof.* Pick the following accessible boundary points (see e.g. [14], Ch II, §3 for definition) of $\partial\Omega$: $-\infty, +\infty, 0$. Then by Theorem 7.2(i) combined with Remark 7.3, there exists a uniquely defined quasi-conformal homeomorphism $f : \Omega \to \mathcal{S}_1$ which preserves these boundary points. Moreover, the smoothness properties required in (iii) follow from Theorem 7.2(ii) and (iii). It remains to prove (6.4).

In view of the periodicity of the domain $\Omega$ the mapping $\tilde{f}(z) = f(z + 2\pi)$ is also a quasi-conformal homeomorphism of $\Omega$ on the straight strip $\mathcal{S}_1$. Note that $\tilde{f}$ sends $-\infty$ and $\infty$ into themselves, and the point 0 into $z_0 = f(2\pi)$. Note that $z_0$ is real, since it lies on the lower portion of the boundary of $\mathcal{S}_1$, i.e. on the straight horizontal line $\operatorname{Im} z = 0$. Consequently, the composition function

$$h(z) = \tilde{f}\big(f^{-1}(z)\big),$$

defined on $\mathcal{S}_1$, is a conformal homeomorphism of $\mathcal{S}_1$ onto itself (see [3], Part II, Sect. 6.2) acting in such a way that $-\infty$, $+\infty$ are preserved and $h(0) = z_0$. It is easy to see that the function $\tilde{h}(z) = z + z_0$ satisfies the same conditions. On the other hand, such a conformal mapping is unique (see e.g. [14], Ch. II, §3, Th. 6). Therefore $h(z) = \tilde{h}(z)$ and hence

$$\tilde{f}(z) = f(z) + \varkappa, \ \ \varkappa = \overline{\varkappa} = z_0.$$

It is clear that $\varkappa \neq 0$, for otherwise the function $f$ would have remained bounded as $|z| \to \infty$, which would contradict the assumption that $f$ sends $\pm\infty$ into itself. For the same reason $\varkappa > 0$, since otherwise $+\infty$ and $-\infty$ would exchange their places under the mapping $f$. □

For conformal mappings the above argument can be found in [14], Ch. V, §1.

Now Theorem 4.2 follows from Theorem 6.2.

6.3. **Bilipschitz mappings of the domain $\Omega$.** In this subsection we address a question which has no direct effect on the results of the paper, but is nevertheless natural and important. If the domain $\Omega$ has no corners or peaks, then according to Theorem 6.2 the homeomorphism $f : \Omega \to \mathcal{S}_1$ is $\mathsf{C}^{1+\alpha}$-smooth. This is guaranteed by the initially assumed $\mathsf{C}^{1+\alpha}$-smoothness of the boundary $\partial\Omega$ (see Condition 2.8) and $\mathsf{C}^{\alpha}$-smoothness of the matrix $\mathbf{G}$ (see (2.19)). In the presence of corners or/and peaks one or both of the derivatives $\partial_z f$, $\partial_z f^{-1}$ are unbounded. In particular, the homeomorphism $f$ may not be Lipshitz even if the boundary $\partial\Omega$ is Lipshitz. Therefore it is legitimate to ask whether a domain $\Omega$ with a Lipshitz boundary $\partial\Omega$ admits a periodic bilipshitz map $\Omega \to \mathcal{S}_1$.

To state the question in a precise form recall that a mapping $F$ from a metric space $(X_1, d_1)$ into a metric space $(X_2, d_2)$ is called *bilipschitz* if there exists a constant $M > 0$ such that

$$d_1(x, y)/M \le d_2(F(x), F(y)) \le M d_1(x, y), \ \ \forall x, y \in X_1.$$

We say that a curve $\ell \in \mathbb{C}$ is Lipshitz if it is a bilipshitz image of $\mathbb{R}$. Note in passing that it is easy to give an intrinsic characterisation of a Lipschitz curve. First of all, it is clear that a Lipschitz curve is Jordan and locally rectifiable. Conversely, let $\ell : \mathbb{R} \to \mathbb{C}$ be a Jordan locally rectifiable curve such that $|\ell(t)| \to \infty$ as $t \to \pm\infty$. Using the arclength parametrisation, one can easily show that $\ell$ is a Lipschitz curve if and only if it is an chord-arc curve, i.e. if there exists a constant $K \ge 1$ such that the length of the subarc of $\ell$ joining any two points is bounded by $K$ times the distance between them.

Assume that the boundary of a periodic domain $\Omega$ consists of two disjoint Lipshitz curves $\ell_+, \ell_-$. Our objective is to find a bilipschitz mapping $F$ of $\mathcal{S}_1$ onto $\Omega$ such that

$$F(z + 2\pi) = F(z) + 2\pi, \ \ \forall z \in \mathcal{S}_1. \tag{6.5}$$

It is evident that the bilipshitz regularity of the curves $\ell_\pm$ is necessary for the existence of such a mapping. Let us convince ourselves that this condition is also sufficient:

**Theorem 6.3.** *Let $\Omega$ be a simply connected periodic domain with a boundary consisting of two disjoint Lipschitz curves $\ell_+$ and $\ell_-$. Then there exists a bilipschitz mapping $F$ of $\mathcal{S}_1$ onto $\Omega$ satisfying* (6.5).

*Proof.* Let $\varphi : \mathcal{S}_1 \to \Omega$ be a conformal homeomorphism, mapping $\pm\infty$ onto $\pm\infty$, 0 onto a given point of $\ell_-$ and such that $\varphi(\zeta + \varkappa) = \varphi(\zeta) + 2\pi$, $\forall \zeta \in \mathcal{S}_1$ with some $\varkappa > 0$. The existence of such a map follows from [14], Ch. V, §1. Since $\ell_\pm$ are Jordan locally rectifiable curves, $\varphi$ can be extended to a homeomorphism of the closure of $\mathcal{S}_1$ onto the closure of $\Omega$ (see [14], Ch. II, §3, Theorem 4), for almost all $x_0 \in \mathbb{R}$ the finite limits

$$\lim_{\tau\to 0+} \varphi'(x_0 + i\tau) \neq 0, \quad \lim_{\tau\to 0+} \varphi'(x_0 + i\pi - i\tau) \neq 0$$

exist, and $\varphi$ is conformal at $x_0$ and $x_0 + i\pi$ (see [14], Ch. X, §1, Theorems 1 and 3). The latter implies that the curves $\ell_\pm$ meet the arcs $\ell_0 = \varphi([x_0, x_0 + i\pi])$ and $\ell_1 = \varphi([x_0 + \varkappa, x_0 + \varkappa + i\pi])$ at the right angle. Now let us define a closed "fundamental domain" of the strip $\mathcal{S}_1$:

$$\Delta := \{x + iy : \ x_0 \le x \le x_0 + \varkappa, \ 0 \le y \le \pi\}.$$

The boundary of $\Delta$ consists of four segments

$$\gamma_- = [x_0, x_0 + \varkappa], \ \gamma_+ = [x_0 + i\pi, x_0 + \varkappa + i\pi],$$
$$\gamma_0 = [x_0, x_0 + i\pi], \ \gamma_1 = [x_0, x_0 + \varkappa + i\pi].$$

The next step is to define a bilipshitz map $F_0$ from $\partial\Delta$ onto $\varphi(\partial\Delta) = \partial\varphi(\Delta)$. Let $\tilde{\ell}_\pm := \varphi(\gamma_\pm) \subset \ell_\pm$. Since $\ell_\pm$ are Lipshitz curves, there exists a bilipshitz homeomorphism $F_0 : \gamma_\pm \to \tilde{\ell}_\pm$. Further, set $F_0(z) = \varphi(z)$ if $z \in \gamma_0$ or $\gamma_1$. In view of the periodicity of $\varphi$, we have

$$F_0(z + \varkappa) = F_0(z) + 2\pi, \ z \in \gamma_0. \tag{6.6}$$

Since the arcs $\tilde{\ell}_+$, $\tilde{\ell}_-$ and $\ell_0$, $\ell_1$ meet at the right angle, it is easy to see that $F_0$ is a bilipschitz mapping of $\partial\Delta$ onto $\partial\varphi(\Delta) = \varphi(\partial\Delta)$. Then it follows from [34] (see also [26], Theorem 7.10 and [22]) that $F_0$ can be extended to a bilipschitz homeomorphism of $\mathbb{C}$ onto itself. This defines a bilipschitz homeomorphism $F_0 : \Delta \to \varphi(\Delta)$. By virtue of (6.6) it is now straightforward to see that the extension $F_1$ of $F_0$, defined by

$$F_1(\zeta + n\varkappa) := F_0(\zeta) + 2\pi n, \quad \zeta \in \Delta, \ n \in \mathbb{Z},$$

is a bilipschitz mapping of $\mathcal{S}_1$ onto $\Omega$ such that $F_1(\zeta + \varkappa) = F_1(\zeta) + 2\pi$, $\forall \zeta \in \mathcal{S}_1$. It remains to define $F$ by

$$F(x + iy) := F_1\left(\frac{\varkappa}{2\pi}\, x + iy\right).$$

□

## 7. General properties of quasi-conformal mappings

We begin with describing a quasi-conformal homeomorphism of the complex plane onto itself. The following general result can be found in [1], [2], [3] Part II, Ch. 6, or [35], Ch. 2.

**Proposition 7.1.** *Let $q \in \mathsf{L}^\infty(\mathbb{R}^2)$ be a function such that $\|q\|_{\mathsf{L}^\infty} < 1$. Then there exists a unique quasi-conformal homeomorphism $f$ of the complex plane such that $f(0) = 0$, $f(2\pi) = 2\pi$ and $f(\infty) = \infty$. Moreover,*

(i) *The conclusions (i), (ii), (iii) of Theorem 6.1 hold;*
(ii) *If $q \in \mathsf{C}^\alpha$, $0 < \alpha < 1$, in a neighbourhood of some point $z_0$, then $f \in \mathsf{C}^{1+\alpha}$ and $|\partial_z f| > 0$ in this neighbourhood.*

The next theorem is a "quasi-conformal" version of Riemann's mapping theorem and the results on the boundary behaviour of conformal mappings. The results collected in this Theorem probably are not new, but we have not been able to find them stated in the form convenient for us.

**Theorem 7.2.** *Let $\Omega \subset \mathbb{C}$ be a simply connected open domain with more than one boundary point, such that all points of $\partial\Omega$ are accessible. Suppose $q \in \mathsf{L}^\infty(\Omega)$ and $\|q\|_{\mathsf{L}^\infty} < 1$. Then*

(i) *There exists a unique $q$–quasiconformal map $w$ of $\Omega$ onto the unit disk $\mathcal{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ which maps three given points $z_1, z_2, z_3 \in \partial\Omega$ indexed in order of their occurrence as one proceeds in the positive direction along $\partial\Omega$ (see* [14]*, Ch. II, §3) onto three given points $\zeta_1, \zeta_2, \zeta_3 \in \partial\mathcal{D}$ similarly indexed. This map and its inverse $f^{-1}$ belong to $\mathsf{W}^{1,\tau}_{\mathrm{loc}}(\Omega)$ for some $\tau > 2$. The map $f$ defines a homeomorphism of the compactification $\widehat{\Omega}$ of $\Omega$ by the accessible points of $\partial\Omega$ onto the closed unit disk.*
(ii) *Let $m \in \mathbb{N}\cup\{0\}$, $0 < \alpha < 1$, $z_0 \in \widehat{\Omega}$, and let $q$ be $\mathsf{C}^{m+\alpha}$–smooth in a neighbourhood of $z_0$. Suppose also that if $z_0 \notin \Omega$ then the intersection of $\partial\Omega$ with a neighbourhood of $z_0$ is a $\mathsf{C}^{m+1+\alpha}$–smooth Jordan curve. Then $f$ is $\mathsf{C}^{m+1+\alpha}$–smooth in a neighbourhood of $z_0$ and $|\partial_z f(z_0)| > 0$.*
(iii) *Let $z_0 \in \partial\Omega$ and suppose that the intersection of $\partial\Omega$ with a neighbourhood of $z_0$ is a piece-wise $\mathsf{C}^{1+\alpha}$–smooth Jordan curve, $0 < \alpha < 1$, with the only angular point at $z_0$, which is not an outward cusp, i.e. the interior with respect to $\Omega$ angle at $z_0$ is nonzero. Let $\gamma : [t_1, t_2] \to \mathbb{C}$, $\gamma(t_0) = z_0$ be a parametrization of this curve in the positive direction. Suppose also that $q$ is $\mathsf{C}^\alpha$–Hölder continuous in a neighbourhood of $z_0$. Then $f$ and its inverse $g := f^{-1}$ satisfy the following conditions*

$$\partial_z f(z) = ((z - z_0) + q(z_0)(\bar{z} - \bar{z}_0))^{\frac{1}{\nu}-1} F(z), \tag{7.1}$$

$$\partial_\zeta g(\zeta) = (\zeta - \zeta_0)^{\nu-1} G(\zeta), \quad \partial_{\bar{\zeta}} g(\zeta) = -q(g(\zeta))\overline{\partial_\zeta g(\zeta)}, \tag{7.2}$$

*where $\zeta_0 = f(z_0)$,*

$$\nu = \frac{1}{\pi} \arg\left\{ -\frac{\gamma'(t_0 - 0) + q(z_0)\overline{\gamma'(t_0 - 0)}}{\gamma'(t_0 + 0) + q(z_0)\overline{\gamma'(t_0 + 0)}} \right\} \in (0, 2], \tag{7.3}$$

*and $F$, $G$ are Hölder continuous and nowhere zero in some neighbourhoods of $z_0$ and $\zeta_0$ respectively.*

*Remark* 7.3. The theorem above uses the disk $\mathcal{D}$ as a "target" domain. This choice was made only for definiteness and convenience of the proof. One can easily restate Theorem

7.2 choosing other simply connected domains as targets. In particular, making obvious modifications, $\mathcal{D}$ can be replaced with the strip $\mathcal{S}_1$. This can be done by mapping $\mathcal{D}$ onto $\mathcal{S}_1$ using a standard conformal map, and noticing that composition of a conformal and a $q$-quasi-conformal mapping is again $q$-quasi-conformal (see [3], Part II, §6.2)

*Proof of Theorem 7.2.* Step I Take an arbitrary extension $q_0 \in \mathsf{L}^\infty(\mathbb{C})$ of $q$ such that $\|q_0\|_{\mathsf{L}^\infty} < 1$. There exists a $q_0$–quasiconformal homeomorphism $w : \mathbb{C} \to \mathbb{C}$ which belongs to $\mathsf{W}^{1,\tau}_{\mathrm{loc}}(\mathbb{C})$ with a $\tau > 2$, together with its inverse (see Proposition 7.1). Let $\Omega_0 := w(\Omega)$, $z'_k := w(z_k)$, $k = 1, 2, 3$. It is clear that $\Omega_0$ is a simply connected domain and all points of $\partial\Omega_0$ are accessible. Hence there exists a unique conformal map $\psi$ of $\Omega_0$ onto $\mathcal{D}$ which maps $z'_1, z'_2, z'_3$ onto $\zeta_1, \zeta_2, \zeta_3$ (see, e.g., [14], Ch. II, §3, Theorem 6). It is not difficult to see that $f := \psi \circ w : \Omega \to \mathcal{D}$ is a $q$–quasiconformal map (see [3], Part II, §6.2) having all the properties announced in (i).

Step II Let $f_1 : \Omega \to \mathcal{D}$ be an arbitrary $q$-quasiconformal homeomorphism mapping $z_1, z_2, z_3$ onto $\zeta_1, \zeta_2, \zeta_3$. Then $f_1 \circ f^{-1} : \mathcal{D} \to \mathcal{D}$ is an analytic (see [3], Part II, §6.2) homeomorphism and hence a conformal automorphism (see [14], Ch. II, §1) with fixed points $\zeta_1, \zeta_2, \zeta_3$. By the uniqueness result for conformal maps we have $f_1 \circ f^{-1}(z) \equiv z$, i.e. $f_1 = f$. This proves uniqueness and shows that the map $f$ constructed above **does not depend on the choice of an extension $\mathbf{q_0}$**.

Step III Under the conditions of (ii) there exists an extension $q_0$ which is $\mathsf{C}^{m+\alpha}$-smooth in a neighbourhood of $z_0$. Then it follows from [35], Theorem 2.9 and the proof of Theorem 2.12, that $w$ from Step I is $\mathsf{C}^{m+1+\alpha}$-smooth in a neighbourhood of $z_0$ and

$$|\partial_z w(z_0)|^2 - |\partial_{\bar z} w(z_0)|^2 > 0. \tag{7.4}$$

So, (ii) will follow if we prove that the conformal map $\psi$ from Step I is $\mathsf{C}^{m+1+\alpha}$-smooth in a neighbourhood of $w(z_0)$ and $\psi'(w(z_0)) \neq 0$. We need to do this only if $w(z_0) \in \partial\Omega_0$, i.e. $z_0 \in \partial\Omega$.

Step IV It follows from Step III that, under the conditions of Part (ii), the intersection of $\partial\Omega_0$ with a neighbourhood of $w(z_0)$ is a $\mathsf{C}^{m+1+\alpha}$-smooth Jordan curve. Let us take a simply connected subdomain $\Omega_1 \subset \Omega_0$ with a $\mathsf{C}^{m+1+\alpha}$-smooth boundary such that $\Gamma := \partial\Omega_1 \bigcap \partial\Omega_0$ is a $\mathsf{C}^{m+1+\alpha}$-smooth subarc of the above Jordan curve, $w(z_0)$ belongs to $\Gamma$ and is different from its endpoints. By the uniqueness result for conformal maps we can choose conformal maps $\psi_1 : \Omega_1 \to \mathcal{D}$ and $\varphi : \mathcal{D} \to \psi(\Omega_1)$ so that

$$\psi = \varphi \circ \psi_1 \quad \text{in} \quad \Omega_1.$$

The map $\psi_1$ is $\mathsf{C}^{m+1+\alpha}$-smooth in the closure of $\Omega_1$ and $\psi_1'(w(z_0)) \neq 0$ (see [26], Theorems 3.5, 3.6)

The mapping $\varphi$ maps an open circular arc containing $\psi_1(w(z_0))$ onto a circular arc. Hence $\varphi$ is holomorphic in a neighbourhood of $\psi_1(w(z_0))$ and $\varphi'(\psi_1(w(z_0))) \neq 0$ (see [14], Ch. II, §3, Theorem 5). Therefore $\psi$ is $\mathsf{C}^{m+1+\alpha}$-smooth in a neighbourhood of $w(z_0)$ and $\psi'(w(z_0)) \neq 0$. Note in passing that this is in fact a variant of Kellogg's theorem (cf., e.g. [21], Sect. 29 or [20], Ch. II, §1, Theorem 1).

The proof of (ii) is now completed.

Step V The proof of (iii) is similar to that of (ii). Let $q_0$ be an extension of $q$ which is $\mathsf{C}^\alpha$-Hölder continuous in a neighbourhood of $z_0$. Then similarly to Step III, $w$ from Step I is $\mathsf{C}^{1+\alpha}$-smooth in a neighbourhood of $z_0$ and it satisfies (7.4). Therefore the intersection of $\partial\Omega_0$ with a neighbourhood of $w(z_0)$ is a piece-wise $\mathsf{C}^{1+\alpha}$-smooth Jordan curve with the only singular point at $w(z_0)$. This curve is parameterized by $[t_1, t_2] \ni t \mapsto w(\gamma(t)) \in \mathbb{C}$. The equality $\partial_{\bar z} w(z) = q(z)\partial_z w(z)$ implies

$$\begin{aligned}\frac{dw(\gamma(t_0 \pm 0))}{dt} &= \partial_z w(z_0)\gamma'(t_0 \pm 0) + \partial_{\bar z} w(z_0)\overline{\gamma'(t_0 \pm 0)} \\ &= (\gamma'(t_0 \pm 0) + q(z_0)\overline{\gamma'(t_0 \pm 0)})\partial_z w(z_0).\end{aligned}$$

Therefore the interior with respect to $\Omega_0 = w(\Omega)$ angle at $w(z_0)$ equals $\pi\nu$, where $\nu$ is given by (7.3).

Step VI Now we need to investigate the properties of the conformal map $\psi : \Omega_0 \to \mathcal{D}$. Using the argument from Step IV we can reduce this to the study of a conformal map $\psi_1 : \Omega_1 \to \mathcal{D}$, where $\Omega_1$ is a simply connected domain with a piece-wise $\mathsf{C}^{1+\alpha}$–smooth boundary and the only singular point $w(z_0)$, where the interior with respect to $\Omega_1$ angle equals $\pi\nu > 0$. Applying Warschawski's theorem (see e.g. [35], Theorem 1.9, and [16], Ch. 3, §3) we obtain that $\psi$ and its inverse $\eta := \psi^{-1}$ satisfy the conditions

$$\psi'(w) = (w - w(z_0))^{\frac{1}{\nu}-1}\Psi(w), \quad \eta'(\zeta) = (\zeta - \zeta_0)^{\nu-1}E(\zeta),\ \zeta_0 = \psi(w(z_0)),$$

where $\Psi$ and $E$ are Hölder continuous and nowhere zero in some neighbourhoods of $w(z_0)$ and $\zeta_0$ correspondingly. Now (iii) follows from the formulae $f = \psi \circ w$, $g = w^{-1} \circ \eta$ and

$$\begin{aligned}w(z) - w(z_0) &= (\partial_z w(z_0)(z - z_0) + \partial_{\bar z} w(z_0)(\bar z - \bar z_0))\,\omega(z, z_0) \\ &= ((z - z_0) + q(z_0)(\bar z - \bar z_0))\,\partial_z w(z_0)\,\omega(z, z_0),\end{aligned}$$

where $\omega(\cdot, z_0)$ is Hölder continuous in a neighbourhood of $z_0$ and $\omega(z_0, z_0) = 1$. The second equality in (7.2) follows from [3], Part II, Ch. 6, Appendix, Theorem 3(iv) (see also [1], Ch. I, Sect. C). □

## 8. Acknowledgements

The authors thank M. Birman, T. Suslina and R. Shterenberg for fruitful discussions and constructive remarks on the manuscript. R. Shterenberg's comments allowed us to remove unnecessary restrictions on the coefficients and thereby simplify the statements and proofs of the main results. We are also grateful to T. Suslina and R. Shterenberg for letting us know about their work [29] prior to its publication.

Our thanks go to the referee who pointed out to us reference [9].

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