The Gelfand-Naimark Theorem

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It should be added that this is work that has taken place over some thirty-five years, over which I have worked with many people, notably Bernhard Banaschewski and Joan Wick Pelletier, to whom I owe a great debt of gratitude.

Algebraicisation

One of the great contributions that algebra has made to mathematics in the last decades has been that the ideas of *sheaf theory* allow structure of great subtlety to be arranged over a topological space. There is, however, a problem that arises when one begins to place the ideas of smooth manifolds in this context.

Consider a smooth manifold X , together with its sheaf Ω_X^0 of smooth functions on X . Alongside this, we wish to consider the tangent sheaf T_X of vector fields on X .

Of course, this is just the sheaf of sections of the tangent bundle $\mathfrak{I}_X$. However, proceeding algebraically, this is also the sheaf that assigns to each open subset U the $\Omega_X^0(U)$-module of derivations of the algebra $\Omega_X^0(U)$ of smooth real functions on U.

The problem is that the assignment

$$U \longmapsto \mathrm{Der}_{\text{‘}}(\Omega_X^0(U))$$

is not canonically functorial in U. Although we know that the restriction homomorphisms exist, we are not quite sure *why* they exist.

Algebraicisation

One answer to this problem, known certainly to Boardman, was to invoke *partitions of unity*, thereby localising the problem. However, there is a more basic technique, that once identified has wide application. This uses the fact that for the sheaf Ω_X^0 of smooth functions on X :

> given any $x \in X$ and any open subset $U \ni x$, there exists $p \in \Omega_X^0(X)$ with support in U such that $p_x = 1$.

That is, that the functions needed to establish the *complete regularity* of the smooth manifold X can actually be found inside the sheaf Ω_X^0 of smooth functions on X.

That the sheaf Ω_X^0 of smooth functions on X satisfies this condition follows by applying to the continuous functions, whose existence is guaranteed by the complete regularity of the underlying topological space, techniques used by Boardman to smooth these functions.

It will be seen shortly, however, that in fact this property follows simply from algebraic facts about the ring $\Omega_X^0(X)$ of smooth functions on the smooth manifold X . But first, we note that it solves the problem of making the taking of the module of derivations into one which is functorial on the lattice of open subsets of the manifold.

Algebraicisation

Observe first an important property of any sheaf of modules M_X over the ringed space (X, Ω_X^0). Given any open subset U, any $x \in U$, and any element $a \in M_X(U)$, there exists an element $b \in M_X(X)$ for which $a_x = b_x$. For, we may find $p \in \Omega_X^0(X)$ with support in U such that $p_x = 1$. Then patch $(p|U)a \in M_X(U)$ with zero on the open complement of the support of p , to obtain an element $b \in M_X(X)$ with $a_x = p_x a_x = b_x$. Hence, any local element of M_X is locally equal to a global element of M_X.

Consider now a derivation D of the algebra $\Omega_X^0(U)$ of smooth functions on an open subset U. We assert first that, for any $f \in \Omega_X^0(U)$ and any open subset $V \subseteq U$, $f|V = 0$ implies $Df|V = 0$. It suffices to show that Df has germ zero at each $x \in V$. But, given $x \in V$ we may choose $p \in \Omega_X^0(U)$ with support in V such that $p_x = 1$. Then, noting that $pf = 0$ by the choice of p, we have that $D(pf) = (Dp)f + p(Df) = 0$. Evaluating the germs at $x \in V$, we find that $(Df)_x = 0$.

Suppose now given open subsets $V \subseteq U$. Observe that given any $f \in \Omega_X^0(V)$ there is an open covering (V_a) of V , together with for each a an element $f_a \in \Omega_X^0(U)$ for which $f_a|V_a = f|V_a$. Then, defining $(D|V)f$ locally on this open covering by writing $((D|V)f)|V_a = Df_a|V_a$, it may be seen by the above remarks that this is a well-defined element of $\Omega_X^0(V)$, and that $D|V$ is a derivation of $\Omega_X^0(V)$. Hence, one may indeed construct the Ω_X^0-module T_X of derivations.

Algebraicisation

A similar technique allows the construction of the Ω^0_M-module Ω^1_M of 1-forms as the dual Ω^0_M-module of the tangent module T_M, although this may be constructed directly from Ω^0_M in the usual way. From there, by standard exterior algebra, the de Rham complex

$$\Omega^0_M \to \Omega^1_M \to \Omega^2_M \to \cdots \to \Omega^n_M \to \cdots$$

may be constructed, and de Rham's theorem stated and proved sheaf-theoretically.

The question that arises is whether, from a geometric viewpoint, the presence of the tangent module, as well as the de Rham complex, in considering abstractions of the differential geometric situation is something that is considered essential, or merely desirable.

For the moment, we consider the question of identifying the ringed spaces (X, A_X) that have this property, and we examine the consequences of this requirement, which turn out to be unexpectedly extensive.

It should be noted that in what follows no assumptions are made about the commutativity of the rings concerned. It will, however, emerge that we appear never to stray far at this stage from the commutative context, with certain significant exceptions.

Algebraicisation

DEFINITION. A ringed space (X, A_X) will be said to be:

> *completely regular* provided that for each $x \in X$ and open subset $U \ni x$ there exists $a \in A_X(X)$ with support in U and such that $a_x = 1$;
>
> *paracompact* provided that subordinate to any open covering (U_α) of X there exists a partition of unity $(p_i)_{i \in I}$ lying in the ring $A_X(X)$;
>
> *compact* provided that subordinate to any open covering (U_α) of X there exists a finite partition of unity $(p_i)_{i=1,\ldots,n}$ lying in the ring $A_X(X)$.

It may be seen straightforwardly that for any ringed space (X, A_X), *compact* implies *paracompact* implies *completely regular*.

Observe that a topological space X is completely regular, paracompact, and compact respectively if the ringed space $(X, ‘_X)$ obtained by taking the sheaf $‘_X$ of continuous real functions on X is completely regular, paracompact, and compact. In the case of a smooth manifold X , the ringed space (X, Ω^0_X) is always paracompact, and is compact exactly if the manifold X is compact.

Algebraicisation

THEOREM. A ringed space (X, A_X) is compact if, and only if, the adjoint functors

$$\mathrm{Mod}\,A \rightleftarrows \mathrm{Mod}_X\,A_X$$

of which the coadjoint assigns to each sheaf of modules over the sheaf of rings A_X its module of sections over the ring A of sections of the ringed space (X, A_X) determine an equivalence of categories.

COROLLARY. A topological space X is compact if, and only if, the functors

$$\mathrm{Mod}\ `(X) \rightleftarrows \mathrm{Mod}_X\ `_X$$

determine an equivalence of categories.

CONSTRUCTION. Given a compact ringed space (X, A_X), the continuous mapping

$$\mathrm{Max}\,A \to X$$

that assigns to each maximal ideal of A the unique $x \in X$ by which it is fixed is a quotient that uniquely determines the compact ringed space. Moreover, any such quotient to a compact topological space X determines a compact ringed space over X .

Algebraicisation

DEFINITION. By a *Gelfand ring* A is meant a ring for which, given any distinct maximal right ideals m, m' of A, there exist elements $a, a' \in A$ for which

$$a \notin m, a' \notin m' \text{ and } aAa' = 0.$$

THEOREM. A ring A is Gelfand if, and only if, it is isomorphic to the ring of sections of a compact local ringed space. Moreover, the category of Gelfand rings is canonically equivalent to the dual of the category of compact local ringed spaces.

COROLLARY. For any Gelfand ring A, the category of finitely generated projective A-modules is canonically equivalent to the category of $A_{\mathrm{Max}\,A}$-vector bundles on the compact space Max A.

As particular cases, we recover Swan's theorem for the ring of continuous real (or complex) continuous functions on a compact space X, and for the ring of smooth functions on a compact smooth manifold X.

There are many other results that carry over to Gelfand rings from the case of rings of continuous real functions on a compact space X. However, it should be remarked that a C*-algebra A is a Gelfand ring if, and only if, A is commutative.

Internalisation

The aim of the preceding comments has been to establish the existence of algebraic techniques for constructing representations of rings as rings of sections of compact ringed spaces. Of course, the observation that any commutative C*-algebra A is a Gelfand ring yields the algebraic side of the Gelfand isomorphism:

$$A \to \check{S}_{\mathrm{Max}\,A}(\mathrm{Max}\,A)\ ,$$

which can be extended straightforwardly to an isomorphism of involutive complex algebras.

However, there remains the question of incorporating the normed structure into the algebraic framework, to the solution of which the methods of sheaf theory classically are not particularly conducive. To the rescue of this dilemma comes the realisation that the category of sheaves on any topological space admits an interpretation of constructive mathematics.

It is to this process that we shall give the name *internalisation*, allowing us relate the mathematics of the world that we are representing with that of the world that represents it. Thus, the Gelfand representation of a commutative C*-algebra A allows it to become the sheaf $\check{S}_{\mathrm{Max}\,A}$ of complex numbers in the constructive world of sheaves on its spectrum Max A.

Internalisation

To give a first instance of the application of this methodology, we may indicate the way in which the generalisation of Swan's theorem already encountered may be obtained. Recall first the following result of Kaplansky:

THEOREM. Any finitely generated projective module over a local ring A is necessarily free.

This is a theorem of which the proof can be straightforwardly adapted to be constructively valid. Rewritten in this way, it allows the theorem to be applied in the category of sheaves over any topological space, more generally in any topos. Applied to the Gelfand representation of any Gelfand ring A , it yields:

COROLLARY. Any locally finitely generated locally projective module over the sheaf $A_{\mathrm{Max}\,A}$ is locally finitely free.

Applying the compactness of the Gelfand representation, the canonical equivalence between the category of A-modules and the category of sheaves of $A_{\mathrm{Max}\,A}$-modules yields the generalisation of Swan's theorem already noted.

Internalisation

At the algebraic level, the internalisation of such results is relatively straightforward. It is simply a matter of finding the appropriate constructive form of the concepts involved, allowing a proof to be constructivised. At the analytic level, however, at which the concepts involved are often higher order, more subtlety is often required. Thus,

DEFINITION. By a *seminormed involutive algebra* A is meant an involutive algebra together with a map

$$N : \mathbb{Q}^+ \to \Omega^A$$

that assigns to each positive rational q a subset $N(q)$ of A satisfying the following conditions:

$\exists q \;\; a \in N(q)$	$a \in N(q) \wedge a' \in N(q') \to aa' \in N(qq')$
$a \in N(q) \wedge a' \in N(q') \to a + a' \in N(q + q')$	$a \in N(q) \to a^* \in N(q)$
$a \in N(q') \to \alpha a \in N(qq')$ whenever $\alpha \in N(q)$	$1 \in N(q)$ whenever $q > 1$
$0 \in N(q)$	$a \in N(q) \leftrightarrow \exists q' < q \;\; a \in N(q')$,

for all $a, a' \in A$, for each complex rational α , and for all positive rationals q, q'.

The seminorm on the algebra therefore has properties which are expressed in terms of the open ball $N(q)$ of radius q about the zero element of A for each positive rational q .

Internalisation

In order to define completeness, once again allowance must be made for the constructive context in which we are working, in this case, for the absence of countable choice. Instead of working with Cauchy sequences, we must work instead with Cauchy approximations:

DEFINITION. By a *Cauchy approximation* on a seminormed *-algebra A will be meant a mapping

$$C : \mathbb{N} \to \Omega^{A}$$

which satisfies the following conditions:

$$\forall n \in \mathbb{N}\ \exists a \in A\ a \in C_n\ , \quad \forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n, n' \geq m\ a \in C_n \wedge a' \in C_{n'} \to a - a' \in N(1/k)\ .$$

A Cauchy approximation C on a seminormed *-algebra A will be said to be *convergent* to an element $b \in A$ provided that $\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ a \in C_n \to a - b \in N(1/k)$.

Then the seminormed *-algebra A will be said to be *complete* provided that for each Cauchy approximation C on the algebra A there exists a unique element $b \in A$ to which C converges.

It may be noted that the uniqueness implies that the seminorm on A satisfies the condition that $(\forall q \in \mathbb{Q}^{+}\ a \in N(q)) \to a = 0$ for each $a \in A$ that defines a *norm*.

Internalisation

In the case of the Gelfand representation of a commutative C*-algebra A , a seminorm may be defined on the commutative involutive algebra $A_{\mathrm{Max}\,A}$ by transferring the norm of the C*-algebra A along the Gelfand isomorphism

$$A \to A_{\mathrm{Max}\,A}(\mathrm{Max}\,A)\ .$$

By the compactness of the representation, although the details are not entirely straightforward, it may be shown that with respect to this seminorm the algebra $A_{\mathrm{Max}\,A}$ is necessarily complete. Moreover, it is a commutative C*-algebra in the category of sheaves on Max A in the following sense:

DEFINITION. By a C*-algebra A is meant a complete involutive seminormed algebra over the complex rationals satisfying the condition that:

$$a \in N(q) \ \leftrightarrow \ aa^* \in N(q^2)$$

for each $a \in A$ and each positive rational q .

It may be remarked that since the Gelfand representation of a Gelfand ring yields a local ring, it follows that the commutative C*-algebra $A_{\mathrm{Max}\,A}$ is necessarily isomorphism to the C*-algebra of complex numbers in the category of sheaves on Max A by the Gelfand-Mazur theorem.

Internalisation

In fact, these observations lead to asking the question of whether the Gelfand representation can itself be obtained constructively, thereby allowing it to be internalised in any topos. That this is the case depends algebraically on whether the representation of a Gelfand ring can be obtained constructively, which is in fact the case. For the moment we focus on the technique that can be applied in the case of a commutative C*-algebra.

The situation is similar to that faced by Hakim in relativising the construction of the Grothendieck representation of a commutative ring. In that case, without the techniques of internalisation, the problem was solved with considerable difficulty and a certain amount of brute force. With the benefit of hindsight, it may be dealt with more efficiently by internalising the construction.

Constructively, when localising, instead of considering the prime ideals of a commutative ring A , we need to consider their complements, the primes of A . These primes may be characterised by considering the following *propositional geometric theory* determined by the commutative ring A :

Internalisation

DEFINITION. Given a commutative ring A, consider the propositional geometric theory defined by introducing a proposition

$$a \in P$$

for each $a \in A$, together with the following axioms:

$$true \vdash 1 \in P$$

$$0 \in P \vdash false$$

$$a + b \in P \vdash a \in P \vee b \in P$$

$$ab \in P \dashv\vdash a \in P \wedge b \in P$$

for all $a, b \in A$.

By the *spectrum* Spec A of the commutative ring A will be meant the Lindenbaum locale of this theory.

In the category of sheaves on the spectrum of A, there exists a representation of A by a local ring $A_{\mathrm{Spec}\,A}$ constructed by localising with respect to the generic prime of A, yielding the Grothendieck representation of the commutative ring A.

Internalisation

DEFINITION. Given a commutative C*-algebra A , consider the propositional geometric theory defined by introducing a proposition

$$a \in A(q)$$

for each $a \in A$ and each non-negative rational q , together with the following axioms:

$$true \vdash 1 \in A(q) \qquad \text{whenever} \quad q < 1 \;;$$
$$a \in A(q) \vdash false \qquad \text{whenever} \quad a \in N(q);$$
$$a \in A(q) \vdash a^* \in A(q);$$
$$a + b \in A(r + s) \vdash a \in A(r) \vee b \in A(s);$$
$$a \in A(r) \wedge b \in A(s) \vdash ab \in A(rs);$$
$$ab \in A(rs) \vdash a \in A(r) \vee b \in A(s);$$
$$a \in A(r) \wedge b \in A(s) \vdash aa^* + bb^* \in A(r^2 + s^2);$$
$$a \in A(q) \vdash\dashv \bigvee_{q' > q} a \in A(q') \;.$$

Then the spectrum Max A of the commutative C*-algebra A is given by the Lindenbaum locale of this theory.

Internalisation

In fact, writing

$$a \in P$$

for the proposition $a \in A(0)$, and applying the lattice operations on the positive cone of A , it may be shown that

$$a \in A(q) \vdash\dashv (a - q.1)^{+} \in P$$

is provable in the theory for any $a \in A$ and any non-negative rational q . With a little more manipulation, it may be shown that the theory is equivalent to that given by the following axioms:

$$true \vdash 1 \in P$$

$$0 \in P \vdash false$$

$$a \in P \vdash a^{*} \in P$$

$$a + b \in P \vdash a \in P \vee b \in P$$

$$ab \in P \vdash\dashv a \in P \wedge b \in P$$

$$a \in P \vdash\dashv \bigvee_i a_i \in P \qquad \text{whenever } a \in \overline{\sum \langle a_i \rangle}$$

for all $a, b \in A$ and any $a_i \in A$.

Internalisation

Consider the propositional geometric theory defined by introducing a proposition

$$z \in (r, s)$$

for each pair r, s of complex rationals, together with the following axioms:

$$z \in (r, s) \vdash \mathit{false} \qquad \text{whenever} \quad (r, s) \leq 0\,;$$

$$\mathit{true} \vdash \bigvee_{(r,s)} z \in (r, s)\,;$$

$$z \in (r, s) \vdash z \in (p, q) \vee z \in (p', q') \qquad \text{whenever} \quad (r, s) \lhd (p, q) \vee (p', q')\,;$$

$$z \in (p, q) \wedge z \in (p', q') \vdash z \in (r, s) \qquad \text{whenever} \quad (p, q) \wedge (p', q') \lhd (r, s)\,;$$

$$z \in (r, s) \vdash\dashv \bigvee_{(r', s') \lhd (r,s)} z \in (r', s')\quad ,$$

in which the conditions are to be interpreted in the geometry of the complex rational plane, expressed algebraically.

Then the Lindenbaum locale of the theory is the locale of complex numbers.

Internalisation

Given a commutative C*-algebra A , one may also consider the propositional geometric theory defined by introducing a proposition

$$a \in (r,s)$$

for each $a \in A$ and each complex rational open rectangle (r,s), together with the following axioms:

$$
\begin{array}{ll}
true \vdash 0 \in (r,s) & \text{whenever} \quad 0 \in (r,s), \text{ and} \\
0 \in (r,s) \vdash false & \text{otherwise;} \\
a \in (r,s) \vdash ta \in (tr,ts) & \text{whenever} \quad t > 0, \text{ and} \\
a \in (r,s) \vdash ia \in i(r,s); & \\
a \in (r,s) \wedge a' \in (r',s') \vdash a + a' \in (r+r', s+s'); & \\
true \vdash a \in N(1) & \text{whenever} \quad a \in N(1); \\
a \in (r,s) \vdash a \in (p,q) \vee a \in (p',q') & \text{whenever} \quad (r,s) \lhd (p,q) \vee (p',q'); \\
a \in (r,s) \dashv\vdash \bigvee_{(r',s') \lhd (r,s)} a \in (r',s'); & \\
true \vdash 1 \in (r,s) & \text{whenever} \quad 1 \in (r,s), \text{ and} \\
1 \in (r,s) \vdash false & \text{otherwise;} \\
a \in (r,s) \vdash a^* \in \overline{(r,s)}; & \\
aa' \in (r,s) \dashv\vdash \bigvee_i \; a \in (p_i,q_i) \wedge a' \in (p'_i,q'_i) & \text{whenever} \quad \bigvee_i (p_i,q_i) \times (p'_i,q'_i) = \mu^*(r,s)
\end{array}
$$

.

Internalisation

The Gelfand-Mazur theorem classically expresses that every maximal ideal of a commutative C*-algebra A is the kernel of a unique multiplicative linear functional on A .

In the constructive context, this is exactly expressed by stating that the interpretation

$$a \in A(q) \quad \mapsto \quad \bigvee_{(r,s) \lhd A(q)} a \in (r,s)$$

of the theory of the maximal spectrum of A in the theory of the multiplicative linear functionals on A is an equivalence of theories, hence determines an isomorphism of locales.

The proof may be reduced to the equivalent result in the geometry of the complex plane, namely that in any bounded region of the complex plane, the topology induced by the rational open codisks

$$z - \lambda.1 \in A(q)$$

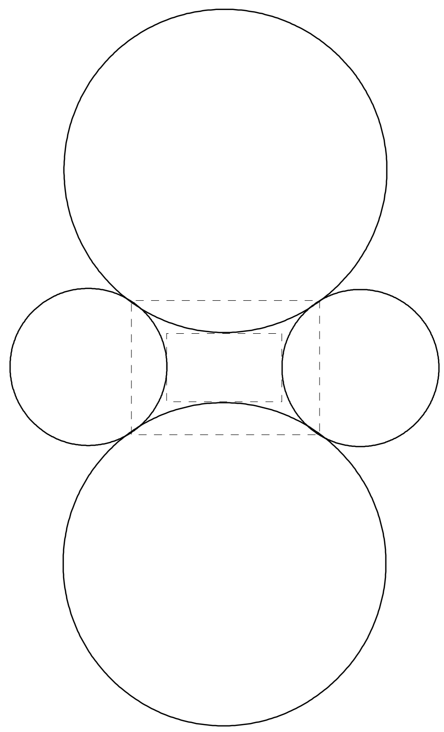
for each complex rational λ and each non-negative rational q coincides with that induced by the rational open rectangles

$$z \in (r,s)$$

for each pair r, s .

Internalisation

The proof is given by the following diagram:



together with the interpretation derived from the inverse assignment

$$z \in (r,s) \quad \mapsto \quad \bigvee z \in A(\lambda_1^q, r_1^q) \wedge \cdots \wedge z \in A(\lambda_4^q, r_4^q) \; .$$

Quantisation

Given a C*-algebra A , consider the propositional geometric theory defined by introducing a proposition

$$a \in P$$

for each $a \in A$, together with the following axioms:

$$\mathit{true} \vdash 1 \in P$$

$$0 \in P \vdash \mathit{false}$$

$$a \in P \vdash a^* \in P^*$$

$$a + b \in P \vdash a \in P \vee b \in P$$

$$ab \in P \dashv\vdash a \in P \,\&\, b \in P$$

$$a \in P \dashv\vdash \bigvee_i a_i \in P \qquad \text{whenever } a \in \overline{\sum \langle a_i \rangle}$$

for all $a, b \in A$ and any $a_i \in A$.

Then the Lindenbaum quantale of this theory is exactly the spectrum of the C*-algebra A .

The Gelfand-Naimark Theorem

In this context, one has the following:

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THEOREM The points of the quantal space Max A are exactly the equivalence classes of irreducible representations of the C*-algebra A .

The Gelfand-Naimark Theorem

In this context, one has the following:

THEOREM The spectrum Max *A* of the C*-algebra *A* is a compact, completely regular quantal space.

THEOREM The points of the quantal space Max *A* are exactly the equivalence classes of irreducible representations of the C*-algebra *A* .

THEOREM The C*-algebra *A* is isometrically *-isomorphic to the C*-algebra of continuous complex functions on the spectrum Max *A* .

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THEOREM The C*-algebra *A* is isometrically *-isomorphic to the C*-algebra of continuous complex functions on the spectrum Max *A* .

In other words, moving to the context of non-commutative involutive intuitionistic logic allows the Gelfand-Naimark theorems to be extended from the commutative to the non-commutative case.

The Gelfand-Naimark Theorem

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In other words, moving to the context of non-commutative involutive intuitionistic logic allows the Gelfand-Naimark theorems to be extended from the commutative to the non-commutative case.

Moreover, this is achieved simply by reinterpreting in the non-commutative context the propositional geometric theory that was used in the commutative case.