The Spectral Theory of Commutative C-Algebras: the Constructive Gelfand-Mazur Theorem*

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INTRODUCTION

The Gelfand-Mazur theorem classically states that any maximal ideal of a commutative C*-algebra A is the kernel of a unique multiplicative linear functional on A. As such, the theorem makes a critical contribution towards proving that the Gelfand representation

$$A \to \mathbb{C}(\mathrm{Max}\, A)$$

of A in the commutative C*-algebra $\mathbb{C}(\mathrm{Max}\, A)$ of continuous complex functions on its maximal ideal space $\mathrm{Max}\, A$ is an isometric *-isomorphism. In turn, this provides a basis for the Gelfand duality which may be established between commutative C*-algebras, always here with unit element, and compact Hausdorff spaces.

This paper, together with its predecessor [3], extend the Gelfand-Mazur theorem to the constructive context of any Grothendieck topos $\mathbb{E}$, and in so doing, along with the constructive form of the Stone-Weierstrass theorem established earlier [2] , provide a basis for extending the Gelfand duality theorem [4] to this context. The motivation for extending these results to that context, together with a discussion of the background definitions and techniques needed, may be found in that predecessor paper [3], to which reference will be made throughout this paper. It will suffice to recall at this stage only that the setting of a Grothendieck topos is wide enough to allow the results to be applied both to C*-bundles, on the one hand, and to C*-algebras acted upon by a group of *-automorphisms, on the other hand.

The form in which the Gelfand-Mazur theorem will be established for any commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$ will be the constructive analogue of that which classically states that the continuous mapping

$$\mathrm{MFn}\, A \to \mathrm{Max}\, A$$

from the space of multiplicative linear functionals on A to the space of maximal ideals of A, defined by assigning to each multiplicative linear functional its kernel, is an isomorphism of compact Hausdorff spaces. Within the constructive context of the Grothendieck topos $\mathbb{E}$, these compact Hausdorff topological spaces are replaced by the compact, completely regular locales determined by the propositional geometric theories which instead describe their classical points, yielding an isomorphism

$$\mathrm{MFn}\, A \to \mathrm{Max}\, A$$

of compact, completely regular locales in the Grothendieck topos $\mathbb{E}$.

It is in this form that the Gelfand-Mazur theorem is later applied [4] to establishing Gelfand duality between the categories of commutative C*-algebras and of compact, completely regular locales in any Grothendieck topos $\mathbb{E}$.

1. THE MAXIMAL SPECTRUM OF A COMMUTATIVE C*-ALGEBRA.

Classically, for any commutative C*-algebra A, one considers its maximal spectrum $\mathrm{Max}\, A$: that is, its set of maximal ideals together with the Zariski topology. A base of open sets for this topology is given by those subsets of the form

$$\{ m \in \mathrm{Max}\, A \mid a \notin m \}$$

for any $a \in A$. Indeed, the maximal spectrum may be considered for any commutative ring A, whether or not it happens to be a C*-algebra. In the case that A is a commutative C*-algebra, the topology may be described equivalently in terms of open sets of the form

$$\{m \in \mathrm{Max}\, A \mid d(a, m) > q \} \ ,$$

in which q denotes any non-negative rational number and $d(a,m)$ denotes the distance of the maximal ideal m from any element $a \in A$. It is a consequence of the lattice-structure existing on the self-adjoint elements of a C*-algebra that these subsets are again open in the Zariski topology. Moreover, since the maximal ideals of A are closed ideals, the Zariski open set determined by any $a \in A$ is exactly the union of those open sets determined by $a \in A$ and all non-negative rationals q. Hence, these subsets actually again form a base for the topology of the maximal spectrum of A. It is this particular way of describing the topology of the maximal spectrum, rather than by the Zariski open sets introduced originally, that proves convenient in the constructive context of considering a commutative C*-algebra in a Grothendieck topos.

To obtain a propositional axiomatisation of the maximal spectrum of a commutative C*-algebra A, one therefore introduces a proposition

$$a \in A(q)$$

for each $a \in A$ and each non-negative rational q. Intuitively, and actually in the classical case, this proposition states that a maximal ideal is distanced strictly by at least the rational q from the element $a \in A$. Classically, the open subset

$$\{ m \in \mathrm{Max}\, A \mid d(a, m) > q \}$$

of the maximal spectrum consists therefore of exactly those maximal ideals that validate this proposition. These subsets of the maximal spectrum of A form a base of open sets for this topology. Moreover, any maximal ideal of the C*-algebra is determined uniquely by those of these subsets to which it belongs (equivalently, those of these propositions which it validates). Hence, we are exactly in the situation in which the locale of maximal ideals of A may be expected to be obtainable constructively by describing the theory which relates these propositions determined by the C*-algebra A .

However, before doing this, it is perhaps important to remark why one chooses an axiomatisation in terms of these propositions, rather than those of the form

$$a \in P$$

for each $a \in A$, of which the intuitive meaning is the assertion of a maximal ideal that it does not contain the element $a \in A$. Classically, it is the Zariski open set

$$\{ m \in \mathrm{Max}\, A \mid a \notin m \}$$

that is the extension of this proposition within the maximal spectrum. Again, these subsets form a base of open sets for its topology, indeed its defining base of open sets. Moreover, these propositions more immediately describe a maximal ideal in terms of those which it validates: the maximal ideal is exactly the complement of the subset consisting of those $a \in A$ for which the corresponding proposition $a \in P$ is validated. It might therefore appear that these propositions might yield a more satisfactory presentation of the theory of the maximal spectrum of a commutative C*-algebra.

Our preference for these propositions

$$a \in A(q)$$

in defining this theory (Mulvey [10]) is that, while containing all the information carried by the propositions

$$a \in P ,$$

these propositions additionally allow immediate access to the normed structure of the C*-algebra Explicitly, knowing those propositions of this kind that are validated classically by a maximal ideal allows one to define a seminorm on A of which the kernel is the maximal ideal concerned. Indeed, anticipating the Gelfand-Mazur theorem, to describe a maximal ideal of a commutative C*-algebra A is equivalent to describing a seminorm on A of which the kernel is a maximal ideal. It is this particular theory, of a seminorm on A of which the quotient is the algebra of complex numbers, that is extremely straightforward to present in terms of these propositions. It has the further advantage of retaining this access to the normed structure of the quotient algebra which is intuitively being described, allowing the isometricity of the Gelfand representation more readily to be obtained at a later stage.

After presenting the theory, our first concern will be to show that it could equivalently, albeit less intuitively, have been presented in terms of the propositions

$$a \in P$$

for each $a \in A$. Evidently this ought to be the case, since classically these topologies coincide on the maximal spectrum. But the exact reasons for this equivalence are central to the techniques needed in working with the maximal spectrum. Classically, these are equivalent

to observing that the Zariski topology and the weak* topology coincide on the spectrum of a commutative C*-algebra. This is equally the observation that the Zariski topology of the maximal spectrum is actually Hausdorff (or, equivalently, that the maximal spectrum is the hausdorffisation of the prime spectrum of the C*-algebra). The equivalence of the axiomatisation in terms of the propositions

$$a \in A(q)$$

now presented, to one in terms of these propositions $a \in P$ will carry little significance in the present discussion, beyond establishing this connection with the Zariski (or hull-kernel) topology. However, its role in the Gelfand-Mazur theorem (and to an even greater extent in the Stone-Weierstrass theorem [2]) is central. At present, we note further only that the proof that these approaches to the maximal spectrum are indeed equivalent rests on the important fact, already observed, that any C*-algebra admits a lattice-structure on its self-adjoint part. This observation, itself depending on the existence of an absolute value

$$|a| = (aa^*)^{\frac{1}{2}}$$

of each $a \in A$, allows one to conclude that the maximal spectrum is indeed an outcome of the algebraic structure of the C*-algebra, as ought to be the case. Consider, then, the propositional theory $\mathbb{M}\mathrm{ax}\, A$ in the topos $\mathbb{E}$, determined by introducing, for each $a \in A$ and each non-negative rational q , a proposition

$$a \in A(q) ,$$

together with the following axioms:

(A1) $\mathit{true} \vdash 1 \in A(q)$ whenever $q < 1$;

(A2) $a \in A(q) \vdash \mathit{false}$ whenever $a \in N(q)$;

(A3) $a \in A(q) \vdash a^* \in A(q)$;

(A4) $a + b \in A(r+s) \vdash a \in A(r) \vee b \in A(s)$;

(A5) $a \in A(r) \wedge b \in A(s) \vdash ab \in A(rs)$;

(A6) $ab \in A(rs) \vdash a \in A(r) \vee b \in A(s)$;

(A7) $a \in A(r) \wedge b \in A(s) \vdash aa^* + bb^* \in A(r^2 + s^2)$;

(A8) $a \in A(q) \dashv\vdash \bigvee_{q' > q} a \in A(q')$.

Then $\mathrm{Max}\, A$ will denote the locale of propositions of the theory, ordered by provable entailment in the theory, modulo provable equivalence.

Of course, exactly as in the case of the locale of linear functionals on the C*-algebra A this description of $\mathrm{Max}\, A$ is entirely equivalent to defining it by introducing a generator for each $a \in A$ and each non-negative rational q , together with relations identical to those written above, except in so far as the relation $\vdash$ of provable entailment is replaced by the order relation $\leq$. The locale $\mathrm{Max}\, A$ is then that determined by these generators and relations in the topos $\mathbb{E}$. However, our choice of logical, rather than algebraic, notation in defining the locale reflects the origin of the presentation of $\mathrm{Max}\, A$, allowing intuitions concerning seminorms to guide manipulation of the theory.

THEOREM. For any commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$, the maximal spectrum $\mathrm{Max}\, A$ is a compact, completely regular locale.

Proof. The compactness of the locale is provable immediately by the technique remarked earlier in connection with the unit square of the complex plane and the unit ball of the locale of linear functionals of the C*-algebra A.

Consider the propositional theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$ obtained from $\mathbb{M}\mathrm{ax}\, A$ by eliminating the infinitary disjunction from its final axiom: in other words, by replacing (A8) by the axiom

$$\text{(A8')} \quad a \in A(q') \vdash a \in A(q) \qquad \text{whenever } q < q' ,$$

omitting any converse assertion. Evidently, the theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$ is now finitary, hence its locale $\mathrm{Max}_{\mathrm{fin}}\, A$ is compact (indeed, coherent), being the ideal lattice of the distributive lattice obtained by taking the propositions of the theory for generators and its axioms for relations. Because the axioms of this finitary part of $\mathbb{M}\mathrm{ax}\, A$ are already contained in the axioms of $\mathbb{M}\mathrm{ax}\, A$ itself, there is a canonical embedding

$$\mathrm{Max}\, A \to \mathrm{Max}_{\mathrm{fin}}\, A$$

of locales. However, the axiomatisation of the theory $\mathbb{M}\mathrm{ax}\, A$ also allows the definition of a retraction

$$\mathrm{Max}_{\mathrm{fin}}\, A \to \mathrm{Max}\, A$$

of this embedding, by the map of locales determined by the assignment

$$a \in A(q) \mapsto \textstyle\bigvee_{q'>q} a \in A(q')$$

to each proposition generating $\mathbb{M}\mathrm{ax}\, A$ of the proposition of $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$ whose entailment is required to be forced in order to recover $\mathbb{M}\mathrm{ax}\, A$ from its finitary part. Once it has been established that this does indeed yield a map of locales, it is immediate, by the axiom (A8) of $\mathbb{M}\mathrm{ax}\, A$ which has been forced, that it is a retraction of the canonical embedding. In particular, it follows that the locale $\mathrm{Max}\, A$ is compact, being a retract of a compact locale.

To establish that this assignment indeed yields a map of locales, it is necessary only to verify that the assignment takes each axiom of $\mathbb{M}\mathrm{ax}\, A$ to an entailment that is provable in the theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$. This will now be argued in turn for each of the axioms of $\mathbb{M}\mathrm{ax}\, A$:

(A1) : It must be verified that $\mathit{true} \vdash \bigvee_{q'>q} 1 \in A(q')$ is provable in $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$, whenever $q < 1$. But if $q < 1$, then there exists q' with $q < q' < 1$. Then $\mathit{true} \vdash 1 \in A(q')$ by the corresponding axiom in $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$, whence the result.

(A2) : Similarly, to show that $\bigvee_{q'>q} a \in A(q') \vdash \mathit{false}$ in $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$, whenever $a \in N(q)$, observe that $a \in N(q)$ implies $a \in N(q')$ for any $q' > q$. But then $a \in A(q') \vdash \mathit{false}$ by (A2) in the theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$, from which the result follows.

(A3) : A similar argument applies.

(A4) : It must be shown that

$$\textstyle\bigvee_{t>r+s} a+b \in A(t) \vdash \bigvee_{r'>r} a \in A(r') \vee \bigvee_{s'>s} b \in A(s')$$

in the theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$. But, given $t > r+s$, there exist $r' > r$ and $s' > s$ with $t > r'+s' > r+s$. Then $a+b \in A(t) \vdash a+b \in A(r'+s')$ by (A8') of $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$. And $a+b \in A(r'+s') \vdash a \in A(r') \vee b \in A(s')$ by (A4), whence the result.

(A5) : The required deduction is that

$$\textstyle\bigvee_{r'>r} a \in A(r') \wedge \bigvee_{s'>s} b \in A(s') \vdash \bigvee_{t>rs} ab \in A(t)$$

in the theory $\mathbb{M}\mathrm{ax}_{\mathrm{fin}}\, A$. But, given $r' > r$ and $s' > s$, one may take $t = r's' > rs$. Then $ab \in A(t)$, again by (A5) of the finitary theory, which yields the result.

(A6) : By an argument similar to that for (A4) ; and,

(A7) : By an argument similar to that for (A5) . Finally,

(A8) : The proof that $\bigvee_{q'>q} a \in A(q') \dashv\vdash \bigvee_{q'>q} \bigvee_{q''>q'} a \in A(q'')$ is simply tautologous: the disjunctions are evidently over the same subsets of the rationals. This establishes that there is indeed the asserted map of locales, from which it follows that the locale $\mathrm{Max}\, A$ is compact.

Before proving the complete regularity of $\mathrm{Max}\, A$, we first make a number of observations about the theory of the maximal spectrum, showing in particular that it could equally have been expressed in terms of a proposition

$$a \in P$$

for each $a \in A$. This expression will be used from here onward to denote that already written $a \in A(0)$. However, the change is intended to be more than merely symbolic: it expresses instead the intuition that this proposition asserts of a maximal ideal, classically, that $a \in A$ is invertible modulo that ideal. Indeed, our first observation concerning these propositions is that the axioms of the theory $\mathbb{M}\mathrm{ax}\, A$ imply that the following conditions are provable:

(P1) $\mathit{true} \vdash 1 \in P$;

(P2) $0 \in P \vdash \mathit{false}$;

(P3) $a+b \in P \vdash a \in P \vee b \in P$;

(P4) $ab \in P \dashv\vdash a \in P \wedge b \in P$.

Each of these derives from the corresponding axiom of $\mathbb{M}\mathrm{ax}\, A$ by taking the case in which the rationals involved are zero, except the last, which can be argued straightforwardly from (A2) , (A4) , and (A6) .

These conditions satisfied by the propositions $a \in P$ in the theory $\mathbb{M}\mathrm{ax}\, A$ are exactly the axioms of the theory $\mathbb{S}\mathrm{pec}\, A$ of the prime spectrum of the commutative ring A. More precisely, they describe the property of an element $a \in A$ of lying in the complement of a prime ideal of the ring. It may be remarked that the axiomatisation of the theory $\mathbb{M}\mathrm{ax}\, A$ considered originally (Mulvey [10]) introduced both the propositions

$$a \in P$$

for each $a \in A$, together with the axioms (P1) - (P4) given above, and the propositions

$$a \in A(q)$$

for each $a \in A$ and each *positive* rational q, together with the axioms (A2)-(A8) already given, and lastly an axiom

$$\text{(I)} \qquad a \in P \dashv\vdash \bigvee_{q>0} a \in A(q)$$

which relates them. In that presentation, it is clear that the theory describes, actually in the complemented form which is appropriate to a constructive context, a seminorm on the algebra A of which the kernel is a prime ideal. In particular, the axiom (I) intuitively expresses that $a \in A$ is invertible exactly if it is bounded away from zero. Of course, a prime ideal that is the kernel of a seminorm is necessarily closed, and the maximal ideals of a commutative C*-algebra are exactly the closed prime ideals. The axiomatisation of the theory $\mathbb{M}\text{ax}\, A$ presented above just simplifies that originally investigated, by noting that the propositions $a \in P$ can be included amongst the propositions $a \in A(q)$ by extending to the case when q is zero.

An immediate consequence of these observations is that the locale Max A is embedded in the prime spectrum Spec A of the commutative C*-algebra A. For, we have established that the assignment

$$a \in P \mapsto a \in P$$

validates the axioms of $\mathbb{S}\text{pec}\, A$ in the theory $\mathbb{M}\text{ax}\, A$ by inclusion, hence determines an embedding

$$\text{Max}\, A \to \text{Spec}\, A$$

of locales. Our aim now is to show that the propositions

$$a \in P$$

for each $a \in A$ actually generate the theory $\mathbb{M}\text{ax}\, A$. Although it will not be proved at this point, it will be shown later that there exists a map of locales

$$\text{Spec}\, A \to \text{Max}\, A$$

which is a retraction of the canonical embedding. Intuitively, this map is that which observes that the closure of any prime ideal of a commutative C*-algebra is a maximal ideal. In fact, the theory $\mathbb{M}\text{ax}\, A$ may be more correctly considered to be the theory of a closed prime ideal of A. The axiomatisation of this theory most directly associated with this viewpoint may be found amongst the closing remarks of Banaschewski-Mulvey [1] . Of course, the existence of this retraction also gives an alternative proof of the fact that the locale Max A is compact, since Spec A is determined by a finitary theory (cf. Mulvey [9]). However, this approach to this fact involves a consideration of the prime spectrum with which we do not here wish for the moment to be concerned.

The property of the C*-algebra that will be needed in establishing that the propositions $a \in P$ generate the theory $\mathbb{M}\text{ax}\, A$ is that the self-adjoint elements form a lattice-ordered Banach algebra, to which we have already alluded. The assignment to each $a \in A$ of its absolute value

$$|a| = (aa^*)^{\frac{1}{2}}$$

defined earlier allows one to define, for any self-adjoint element $a \in A$, the elements

$$a^+ = \tfrac{1}{2}(|a| + a)$$

and

$$a^- = \tfrac{1}{2}(|a| - a) ,$$

that are respectively its positive and negative parts, in the sense that each is non-negative and that

$$a = a^+ - a^- .$$

It may be remarked in passing that, of course,

$$a^+ a^- = 0$$

for any self-adjoint element $a \in A$.

An immediate observation that can be made is that:

$$a \in A(q) \dashv\vdash |a| \in A(q)$$

is provable in the theory $\mathbb{M}\text{ax}\, A$. For one has that $a \in A(q) \vdash aa^* \in A(q^2) \vdash |a|^2 \in A(q^2)$ $\vdash |a| \in A(q)$, respectively by (A3) and (A5), the equality of aa^* and $|a|^2$, and (A6), with the converse proved similarly. Hence, one need only consider those propositions

$$a \in A(q)$$

for which $a \in A$ is self-adjoint and non-negative. It is recalled that the subset of the commutative C*-algebra A consisting of these elements is denoted A^+ in the topos. Observe that for any $a, b \in A^+$, and any positive rationals r, s, one has that:

$$a \in A(r) \wedge b \in A(s) \vdash a + b \in A(r+s) .$$

In a sense, it is this that the axiom (A7) is seeking to express, without explicit reference being made to non-negative elements of the algebra A. Deducing this fact from (A7), suppose that $a, b \in A^+$ are given. Then, for any $r' > r$ and $s' > s$, there exist p, q with $r' > p^2 > r$ and $s' > q^2 > s$; hence, $a \in A(r') \vdash a \in A(p^2) \vdash a^{1/2} \in A(p)$, and $b \in A(s') \vdash b \in A(q^2)$ $\vdash b^{1/2} \in A(q)$, by (A8) and (A6). Then, applying (A7) and (A8), one has that $a \in A(r') \wedge b \in A(s') \vdash a + b \in A(p^2 + q^2) \vdash a + b \in A(r+s)$. But, again by (A8), $a \in A(r)$ $\vdash \bigvee_{r'>r} a \in A(r')$, and $b \in A(s) \vdash \bigvee_{s'>s} b \in A(s')$, from which it follows that $a \in A(r) \wedge b \in A(s) \vdash a + b \in A(r+s)$, as required. This remark enables us, in turn, to show that, for any $a \in A^+$, and any positive rationals r, s :

$$a \in A(r) \vdash a + s.1 \in A(r+s) .$$

For, given $r' > r$, there exists $s' < s$ with $r + s < r' + s'$. Then, by the observation above, together with (A8), one has that $a \in A(r') \vdash a + s.1 \in A(r' + s') \vdash a + s.1 \in A(r+s)$, since *true* $\vdash s.1 \in A(s')$ by (A1), (A2) and (A6), and $r + s < r' + s'$ yields the last entailment by (A8). But, $a \in A(r) \vdash \bigvee_{r'>r} a \in A(r')$ by (A8) once again yields that $a \in A(r)$ $\vdash a + s.1 \in A(r+s)$, as asserted.

It can be shown that, given any $a \in A^+$, and any positive rational q, the proposition

$$a \in A(q)$$

is provably equivalent in the theory $\mathbb{M}\mathrm{ax}\,A$ to the proposition

$$(a-q.1)^+ \in P\ .$$

For, given any positive rational r, one may argue the following entailments:

$$\begin{aligned}(a-q.1)^+ \in A(r) \quad &\vdash q.1+(a-q.1)^+ \in A(q+r)\\ &\vdash a+(a-q.1)^- \in A(q+r)\\ &\vdash a \in A(q) \vee (a-q.1)^- \in A(r)\\ &\vdash a \in A(q)\ ,\end{aligned}$$

respectively by our preceding remarks, the fact that $a-q.1$ equals $(a-q.1)^+ - (a-q.1)^-$, (A4), and the fact that $(a-q.1)^+(a-q.1)^-$ is zero, which implies that $(a-q.1)^- \in A(r) \vdash$ *false*, by (A2) and (A5). But, again by (A8), one has that $(a-q.1)^+ \in A(0) \vdash \bigvee_{r>0} (a-q.1)^+ \in A(r)$. Hence, by the above,

$$(a-q.1)^+ \in P \vdash a \in A(q)\ .$$

Conversely, $a \in A(q) \vdash a-(a-q.1)^+ \in A(q) \vee (a-q.1)^+ \in A(0) \vdash (a-q.1)^+ \in A(0)$ by (A4) and the fact that $0 \le a-(a-q.1)^+ \le q.1$ implies $a-(a-q.1)^+ \in A(q) \vdash$ *false*, by (A2). Thus,

$$a \in A(q) \vdash (a-q.1)^+ \in P\ .$$

Therefore, one concludes that the equivalence

$$a \in A(q) \dashv\vdash (a-q.1)^+ \in P$$

is provable in the theory $\mathbb{M}\mathrm{ax}\,A$ for any element $a \in A$ and any non-negative rational q, by the earlier remark that the elements of A involved can be assumed self-adjoint and non-negative without loss of generality. Intuitively, it is this remark that shows that the weak* topology on the maximal spectrum is equivalent to the Zariski topology, hence, that the maximal spectrum is necessarily Hausdorff for a commutative C*-algebra.

The consequence that we now have concerning the theory $\mathbb{M}\mathrm{ax}\,A$ is that any proposition may be expressed in terms of those of the form

$$a \in P\ ,$$

in which, moreover, $a \in A$ may be taken to be a non-negative self-adjoint element of the C*-algebra A. Explicitly, any proposition of $\mathbb{M}\mathrm{ax}\,A$ is provably equivalent to a disjunction of finite conjunctions of propositions of this form. Moreover, since these propositions satisfy the axioms (P1) - (P4) of the theory $\mathbb{S}\mathrm{pec}\,A$, any finite conjunction of propositions of this kind is again such a proposition. So, any proposition of $\mathbb{M}\mathrm{ax}\,A$ may be expressed, to within provable equivalence, as an arbitrary disjunction of these propositions. In other words, the elements of the locale $\mathrm{Max}\,A$ determined by the propositions $a \in P$ for $a \in A$ form a base for the locale. It is this remark that will prove important in establishing both the Gelfand-Mazur theorem and the Stone-Weierstrass theorem, each of which depends critically on the hausdorffness, as it were, of the maximal spectrum. For the moment, however, it is only some of the results intermediate in obtaining this equivalence that will be needed in demonstrating the complete regularity of the locale $\mathrm{Max}\,A$.

The main step towards establishing this complete regularity is to show that

$$a \in A(s) \lhd a \in A(r)$$

in the locale $\mathrm{Max}\,A$, whenever $r < s$. It will be recalled that this property of the element $a \in A(s)$ being rather below the element $a \in A(r)$ in the locale may be established, in particular, by showing that the join of $a \in A(r)$ with the pseudocomplement $\neg(a \in A(s))$ of $a \in A(s)$ is the identity element of the locale $\mathrm{Max}\,A$. In terms of the theory $\mathbb{M}\mathrm{ax}\,A$, this corresponds to proving, in a language temporarily extended to include the negation of any proposition, that

$$\mathit{true} \vdash a \in A(r) \vee \neg(a \in A(s))$$

is provable from the axioms of the theory. To prove this, note firstly that it may, as usual, be assumed that the element $a \in A$ is self-adjoint and non-negative. Then, observe that $1+a \in A$ is invertible, by applying the observation that $1+bb^* \in A$ is invertible, in any commutative C*-algebra A, to the square root of $a \in A^+$. By a straightforward argument, it follows that the element $a+q.1 \in A$ is invertible for any positive rational q.

Now, given any $r < s$, choose r', s' with $r < r' < s' < s$, together with a positive rational q with $q < r'-r$ and $q < s-s'$. Then $r'/s' < 1$ implies that $\mathit{true} \vdash 1 \in A(r'/s')$ is provable, by (A1). Applying (A6) to the product of the invertible element $a+q.1 \in A$ with its inverse, we obtain a proof of

$$\mathit{true} \vdash a+q.1 \in A(r') \vee 1/(a+q.1) \in A(1/s')\ .$$

The required assertion in the extended language is obtained by showing that from the first of these propositions in this disjunction may be deduced $a \in A(r)$, while the second entails $\neg(a \in A(s))$. For, $a+q.1 \in A(r') \vdash a \in A(r) \vee q.1 \in A(r'-r)$, by (A4); however, $q.1 \in A(r'-r) \vdash$ *false* by (A2) and the choice of $q < r'-r$. So,

$$a+q.1 \in A(r') \vdash a \in A(r)$$

is provable. Similarly, $a \in A(s) \vdash a+q.1 \in A(s') \vee q.1 \in A(s-s')$, while $q.1 \in A(s-s') \vdash$ *false*. So $a \in A(s) \vdash a+q.1 \in A(s')$. But, in case $1/(a+q.1) \in A(1/s')$ we may deduce that $a \in A(s) \vdash 1 \in A(1)$ by (A5), whereas $1 \in A(1) \vdash$ *false*, by (A2). So,

$$1/(a+q.1) \in A(1/s') \vdash \neg(a \in A(s))\ .$$

Combining these entailments with the disjunction proved earlier, we conclude that

$$\mathit{true} \vdash a \in A(r) \vee \neg(a \in A(s))$$

is provable whenever $r < s$, from which the assertion that

$$a \in A(s) \lhd a \in A(r)$$

in the locale $\mathrm{Max}\,A$ follows in the manner indicated.

Now, to show the complete regularity of $\mathrm{Max}\,A$ it is enough to prove that each basic proposition

$$a \in P$$

of the locale is the join of elements that are completely below it. Remarking that

$$a \in P \mapsto \bigvee_{q>0} a \in A(q) \ ,$$

by the axiom (A8) of the theory $\mathbb{M}\mathrm{ax}\, A$, it clearly suffices to prove that

$$a \in A(q) \lhd\!\!\lhd\, a \in P$$

in the locale Max A for each $q > 0$. But from the remark that

$$a \in A(s) \lhd a \in A(r)$$

whenever $r < s$, it follows that the family $a \in A(kq/2^i)$, for $i = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^i$ depending on i , provides an interpolation (actually reversed) from $a \in P$ to $a \in A(q)$, satisfying all the conditions that that requires. The locale Max A is therefore completely regular, completing the proof of the theorem.

Given the fact that Max A is a compact, completely regular locale, together with the observation that its points are exactly the maximal ideals of the C*-algebra A , provided that the topos concerned is one in which the Axiom of Choice is satisfied, one now has the following:

Corollary. For any commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$ in which is satisfied the Axiom of Choice, the maximal spectrum Max A is exactly the Zariski topology of the maximal ideal space of A .

Proof. It has already been established that the points of Max A are given exactly by the maximal ideals of the commutative C*-algebra A . Moreover, the topology of its space of points is that determined by the propositions

$$a \in P$$

which have been shown to generate the theory $\mathbb{M}\mathrm{ax}\, A$. Hence, the topological space of points of the locale Max A is exactly the maximal ideal space of A in the Zariski topology. The corollary now follows, by noting that, in the presence of the Axiom of Choice, the compact, completely regular locale Max A is exactly the topology of its space of points.

It may be remarked in conclusion that although we have been considering throughout a commutative C*-algebra A , the proof of the compactness of the locale Max A remains valid for any seminormed algebra A . The complete regularity of the locale, expressing constructively what classically would be the hausdorffness of the maximal spectrum, requires rather more of the algebra A . Indeed, in the case of a commutative Banach algebra A , classically,the condition that the weak* topology on the spectrum coincides with the Zariski topology, thereby making the spectrum Hausdorff, is described by saying that A is a regular Banach algebra. Although the axiomatisation of Max A is not intended to describe the maximal spectrum of an arbitrary commutative Banach algebra A , there is a generalisation of the concept of commutative C*-algebra to which this construction, together with the proof of the complete regularity of the locale which results, may be extended.

A careful examination of the proofs involved in establishing the complete regularity of the maximal spectrum reveals that one requires only a seminormed algebra which satisfies the conditions of the following:

Definition. By a *commutative pre-C*-algebra* A will be meant a commutative seminormed *-algebra over the complex rationals in the topos $\mathbb{E}$ for which:

i) $aa^* \in N(q^2) \rightarrow a \in N(q)$ for each $a \in A$ and each positive rational q ;

ii) $(aa^*)^{1/2} \in A$ exists for each $a \in A$;

iii) $1 + aa^* \in A$ is invertible for each $a \in A$.

It may be remarked that the first of these conditions guarantees that the completion of the seminormed *-algebra will necessarily be a commutative C*-algebra. The other conditions allow one to conclude that the algebra concerned inherits from its completion the existence of the square root of any non-negative element $aa^* \in A$ and of the inverse of any element of the form $1 + aa^* \in A$. Hence, in particular, any commutative pre-C*-algebra has an absolute value

$$|a| = (aa^*)^{1/2}$$

which endows it with the same lattice-theoretic properties as a commutative C*-algebra.

With these remarks, the observations made above then yield the following:

Corollary. For any commutative pre-C*-algebra A in a Grothendieck topos $\mathbb{E}$, the maximal spectrum Max A is a compact, completely regular locale.

The importance of this observation will lie in the fact that the inverse image of a commutative C*-algebra A along any geometric map

$$\gamma : \mathbb{F} \rightarrow \mathbb{E}$$

will satisfy the conditions concerned, allowing the complete regularity of its maximal spectrum to be concluded. For the inverse image $\gamma^* A$ is certainly a seminormed algebra in the topos $\mathbb{F}$, noting that this is only taken to include that it is an algebra over the rational complex field in $\mathbb{F}$, rather than the complex field, for which the completeness of the algebra would be required. It is seminormed by the open balls which are the inverse images of those in the topos $\mathbb{E}$, the inverse image of the open ball $N(q)$ of A of radius q being denoted by $N^*(q)$. The seminormed structure of the algebra $\gamma^* A$ is determined entirely by these open balls, indexed by rationals originating in the topos $\mathbb{E}$, because the rationals in the topos $\mathbb{F}$ are obtained by pulling these back along the geometric map. For this reason, it follows that the inverse image $\gamma^* A$ satisfies the condition

$$aa^* \in N^*(q^2) \rightarrow a \in N^*(q)$$

required of the norm of a commutative C*-algebra. In particular, the completion

$$\gamma^* A \rightarrow B$$

of the inverse image is therefore a commutative C*-algebra in the topos $\mathbb{F}$.

The other conditions defining a commutative pre-C*-algebra in the topos $\mathbb{F}$ are equally satisfied by the inverse image $\gamma^* A$, since the conditions concerned are each expressible in geometric form, hence are preserved under inverse image functors. Indeed, one may observe more generally that the inverse image of any commutative pre-C*-algebra is again a commutative pre-C*-algebra. The concept concerned may therefore be considered to be the geometric part of that of a commutative C*-algebra.

Considering for each commutative C*-algebra A in the topos $\mathbb{E}$, the completion

$$\gamma^* A \to B$$

of its inverse image in the topos $\mathbb{F}$ allows the inverse image functor of a geometric map

$$\gamma : \mathbb{F} \to \mathbb{E}$$

to determine a functor from the category of commutative C*-algebras in $\mathbb{E}$ to that of commutative C*-algebras in $\mathbb{F}$. Moreover, the maximal spectrum of the commutative C*-algebra B in $\mathbb{F}$ thus obtained may be shown to be canonically isomorphic to that of the inverse image of the commutative C*-algebra A . In particular, the maximal spectrum of the commutative C*-algebra B is exactly that obtained by taking the inverse image of the theory of the maximal spectrum of A .

2. The Gelfand-Mazur theorem.

With these preliminaries, we may now turn to establishing the result which provides the foundation on which the Gelfand representation, and the duality to which it leads, may be based. Classically, for any commutative C*-algebra A , the space of multiplicative linear functionals on A may be identified with the space of maximal ideals of A , by the mapping which assigns to each functional the maximal ideal which is its kernel. The inverse image of the subbasic open set

$$\{ \, m \mid d(a, m) > q \, \}$$

is then the subset

$$\{ \, \varphi \mid |\varphi(a)| > q \, \}$$

of the space of multiplicative linear functionals of A .

Since the spaces concerned are each compact Hausdorff, establishing the fact that this mapping is a homeomorphism amounts precisely to showing that each maximal ideal of A is the kernel of a unique multiplicative linear functional on A . It is this fact which may be viewed as the ultimate content of the Gelfand-Mazur theorem. Indeed, it is in this form that the Gelfand-Mazur theorem will now be proved in the context of a Grothendieck topos $\mathbb{E}$.

So consider the map

$$\mathrm{MFn}\, A \to \mathrm{Max}\, A$$

from the locale MFn A of multiplicative linear functionals on A , of which details of the construction may be found in the predecessor [3] to this paper, to the locale Max A of maximal ideals of A , defined in the following way: the inverse image mapping of the map of locales is defined by assigning to each proposition $a \in A(q)$ which generates the locale Max A the proposition $a \in A(q)$ of the theory $\mathbb{M}$Fn A . It may be recalled that $a \in A(q)$ denotes the disjunction

$$\bigvee a \in (r,s) \quad ((r,s) \lhd A(q))$$

indexed by those rational open intervals (r,s) which are rather below the open codisc $A(q)$ of radius q in the locale of complex numbers. To show that this assignment determines a map of locales, it must be proved that each axiom of the theory $\mathbb{M}$ax A is satisfied by this interpretation in the locale MFn A . But, since MFn A is the locale of the theory $\mathbb{M}$Fn A , this means simply checking that each axiom of the theory $\mathbb{M}$ax A is provable in the theory $\mathbb{M}$Fn A , after interpretation by the assignment

$$a \in A(q) \mapsto a \in A(q)$$

defined above.

This will be argued in turn for each of the axioms of $\mathbb{M}$ax A :

(A1) : It must be proved in $\mathbb{M}$Fn A that

$$\mathit{true} \vdash \bigvee_{(r,s) \lhd A(t)} 1 \in (r,s) \quad \text{whenever } t < 1 \, .$$

But, given $t < 1$, one can choose $(r,s) \lhd A(t)$ such that $1 \in (r,s)$, and then $\mathit{true} \vdash 1 \in (r,s)$ by (M7) .

(A2) : It must be proved in $\mathbb{M}$Fn A that

$$\bigvee_{(r,s) \lhd A(t)} a \in (r,s) \vdash \mathit{false} \quad \text{whenever } a \in N(t) \, , \, 0 < t \, ,$$

i.e. that $a \in (r,s) \vdash \mathit{false}$ whenever $(r,s) \lhd A(t)$. Now, $a \in N(t)$ implies $\mathit{true} \vdash a \in N(t)$ in $\mathbb{M}$Fn A by (M4) and (M2) , since $a \in N(t)$ implies $a/t \in N(1)$ in A . Further, for each $(p,q) \lhd N(t)$, $a \in (p,q) \wedge a \in (r,s) \vdash 0 \in (p-s, q-r)$ by (M2) and (M3) , and $0 \in (p-s, q-r) \vdash \mathit{false}$ by (M1) , because $(p,q) \leq N(t)$ and $(r,s) \leq A(t)$ implies (p,q) and (r,s) are disjoint and therefore $0 \notin (p-s, q-r)$. Since $a \in N(t)$ in MFn A is actually $\bigvee_{(p,q) \lhd N(t)} a \in (p,q)$, this says that $a \in N(t) \wedge a \in (r,s) \vdash \mathit{false}$, and hence that $a \in (r,s) \vdash \mathit{false}$.

(A3) : It must be proved in $\mathbb{M}$Fn A that

$$\bigvee_{(p,q) \lhd A(r)} a \in (p,q) \vdash \bigvee_{(u,v) \lhd A(r)} a^* \in (u,v) \, .$$

This follows directly from (M8) and the fact that $(p,q) \lhd A(r)$ implies that $\overline{(p,q)} \lhd A(r)$ because $A(r)$ is stable under complex conjugation.

(A4) : It must be proved in $\mathbb{M}$Fn A that

$$\bigvee_{(p,q) \lhd A(r+s)} a+b \in (p,q) \vdash \bigvee_{(p',q') \lhd A(r)} a \in (p',q') \vee \bigvee_{(p'',q'') \lhd A(s)} b \in (p'',q'') \, .$$

To begin with, note that, for rational $t > r$ and $a \in A$, $\mathit{true} \vdash a \in A(r) \vee a \in N(t)$ in $\mathbb{M}$Fn A . To see this, observe firstly that any given (u,v) can be covered by dividing each side into n equal parts and then taking, for each vertex z_i of the resulting grid, the open rectangle (p_i, q_i) determined by its vertically neighbouring vertices. For n such that

$(u_1 - v_1)^2 + (u_2 - v_2)^2 < (n/3)^2(t-r)^2$, it is then obvious that $(p_i, q_i) \lhd N(t)$ or $(p_i, q_i) \lhd A(r)$, and by (M5) it follows that $a \in (u, v) \vdash a \in N(t) \vee a \in A(r)$. Now take k such that $a \in N(k)$ in A . Then $\mathit{true} \vdash a \in N(k)$ in $\mathbb{M}\mathrm{Fn}\, A$, as observed before, and consequently $\mathit{true} \vdash a \in N(t) \vee a \in A(r)$, as claimed. Now, given $(p, q) \lhd A(r+s)$, take $w > r+s$ and $t > r$ such that $w-(r+s) > t-r$ and hence also $w-t > s$. Then, for any $(u, v) \lhd N(t)$, one has $a \in (u, v) \wedge a+b \in (p, q) \vdash b \in (p-v, q-u)$ by (M2) and (M3) , and, since $(p-v, q-u) \leq A(w-t)$ by the triangle inequality in $\mathbb{C}$, and $A(w-t) \lhd A(s)$, it follows that $a \in (u, v) \wedge a+b \in (p, q) \vdash b \in A(s)$. This shows that $a \in N(t) \wedge a+b \in A(r+s) \vdash b \in A(s)$, and therefore that $a+b \in A(r+s) \vdash a \in A(r) \vee b \in A(s)$, by the fact that $\mathit{true} \vdash a \in A(r) \vee a \in N(t)$.

(A5) : It must be proved in $\mathbb{M}\mathrm{Fn}\, A$ that

$$\bigvee_{(p,q) \lhd A(r)} a \in (p, q) \wedge \bigvee_{(p',q') \lhd A(s)} b \in (p', q') \vdash \bigvee_{(p'',q'') \lhd A(rs)} ab \in (p'', q'') .$$

If $(p, q) \lhd A(r)$ and $(p', q') \lhd A(s)$, then, by the continuity of the multiplication of complex numbers, each of (p, q) and (p', q') is a finite union of open rectangles (p_i, q_i) and (p'_j, q'_j) , respectively, such that there exists, for each pair i, j an $(l_{ij}, m_{ij}) \lhd A(rs)$ with $(p_i, q_i) \cdot (p'_j, q'_j) \lhd (l_{ij}, m_{ij})$. Then, $a \in (p, q) \wedge b \in (p', q') \vdash \bigvee_{i,j} a \in (p_i, q_i) \wedge b \in (p'_j, q'_j)$ and $a \in (p_i, q_i) \wedge b \in (p'_j, q'_j) \vdash ab \in (l_{ij}, m_{ij})$ by (M5) and (M9) , and therefore $a \in (p, q) \wedge b \in (p', q') \vdash ab \in A(rs)$.

(A6) : It must be proved in $\mathbb{M}\mathrm{Fn}\, A$ that

$$\bigvee_{(p,q) \lhd A(rs)} ab \in (p, q) \vdash \bigvee_{(p',q') \lhd A(r)} a \in (p', q') \vee \bigvee_{(p'',q'') \lhd A(s)} b \in (p'', q'') .$$

For any r and s , if $r' > r$ and $s' > s$, then $\mathit{true} \vdash a \in A(r) \vee a \in N(r')$ and $\mathit{true} \vdash b \in A(s) \vee b \in N(s')$, hence $\mathit{true} \vdash a \in A(r) \vee b \in A(s) \vee (a \in N(r') \wedge b \in N(s'))$ by conjunction. Further, $a \in N(r') \wedge b \in N(s') \vdash ab \in N(r's')$ by (M9) applied to open rectangles covering $N(r')$ and $N(s')$, and essentially the first part of the argument for (A5) . As a result, $\mathit{true} \vdash a \in N(r) \vee b \in N(s) \vee ab \in N(r's')$. Now, given $(p, q) \lhd A(rs)$, choose $r' > r$ and $s' > s$, such that, still, $(p, q) \lhd A(r's')$. Then, $(p, q) \wedge N(r's')$ is empty, and therefore $ab \in (p, q) \wedge ab \in N(r's') \vdash \mathit{false}$. Combining this with the previous result yields $ab \in (p, q) \vdash a \in A(r) \vee b \in A(s)$ for all $(p, q) \lhd A(rs)$, which establishes the desired conclusion.

(A7) : It must be proved in $\mathbb{M}\mathrm{Fn}\, A$ that

$$\bigvee_{(p,q) \lhd A(r)} a \in (p, q) \wedge \bigvee_{(p',q') \lhd A(s)} b \in (p', q') \vdash \bigvee_{(p'',q'') \lhd A(r^2+s^2)} aa^* + bb^* \in (p'', q'') .$$

By the continuity of the multiplication of the complex numbers each $(p, q) \lhd A(r)$ is a finite join of open rectangles (p_i, q_i) , for each of which there exists an $(l_i, m_i) \lhd A(r^2)$ such that $(p_i, q_i) \cdot \overline{(p_i, q_i)} \lhd (l_i, m_i)$. Then $a \in (p, q) \vdash \bigvee_i a \in (p_i, q_i)$ by (M5) , and $a \in (p_i, q_i) \vdash aa^* \in (l_i, m_i)$ by (M8) and (M9) , and therefore $a \in A(r) \vdash aa^* \in A(r^2)$. In the same manner, $b \in A(s) \vdash bb^* \in A(s^2)$, and then the desired result $a \in A(r) \wedge b \in A(s) \vdash aa^* + bb^* \in A(r^2 + s^2)$ follows, by (M3) applied to open rectangles covering $A(r^2)$ and $A(s^2)$, and the fact that $(u, v) \lhd A(r^2)$ and $(u', v') \lhd A(s^2)$ together imply that $(u+u', v+v') \lhd A(r^2 + s^2)$.

(A8) : It must be proved in $\mathbb{M}\mathrm{Fn}\, A$ that

$$\bigvee_{(p,q) \lhd A(r)} a \in (p, q) \dashv\vdash \bigvee_{r<s} \bigvee_{(u,v) \lhd A(s)} a \in (u, v) .$$

This is a direct application of (M6) and the facts that, firstly, $(p, q) \lhd A(r)$ implies that there exist $s > r$ for which, still, $(p, q) \lhd A(s)$, and, secondly, $r < s$ and $(u, v) \lhd A(s)$ imply $(u, v) \lhd A(r)$, which completes the verification.

The assignment therefore determines a map of locales

$$\mathrm{MFn}\, A \to \mathrm{Max}\, A ,$$

corresponding classically to the mapping which assigns to each multiplicative linear functional on A the maximal ideal of A which is its kernel. Concerning this map of locales, we may now prove the following form of the Gelfand-Mazur theorem:

THEOREM For any commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$, the canonical map of locales

$$\mathrm{MFn}\, A \to \mathrm{Max}\, A$$

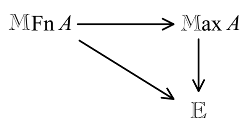
is an isomorphism.

Proof. The assertion will be proved by showing that the inverse image mapping of this map of locales is both injective and surjective. Its injectivity will be established by considering a Barr covering of the topos $\mathbb{E}$, while its surjectivity will be argued constructively within the topos $\mathbb{E}$.

Observe firstly that the map of locales determines a geometric morphism

$$\mathbb{M}\mathrm{Fn}\, A \to \mathbb{M}\mathrm{ax}\, A$$

from the topos of sheaves in $\mathbb{E}$ on the locale $\mathrm{MFn}\, A$ to the topos of sheaves in $\mathbb{E}$ on the locale $\mathrm{Max}\, A$, making the diagram



commute. Proving that the map of locales is a quotient map, that is, that its inverse image mapping is injective, is equivalent to showing that this geometric morphism is a covering.

Consider then a covering

$$\gamma : \mathbb{B} \to \mathbb{E}$$

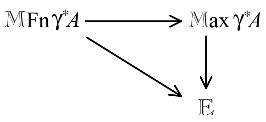
of the Grothendieck topos $\mathbb{E}$ by a topos $\mathbb{B}$ in which the Axiom of Choice (AC) is satisfied, of which the existence follows from a theorem of Barr (Barr [5], Diaconescu [6], Johnstone [8]). The inverse image $\gamma^* A$ of the seminormed algebra A in the topos $\mathbb{E}$ is again a seminormed algebra in the topos $\mathbb{B}$, allowing a map

$$\mathrm{MFn}\ \gamma^*A \to \mathrm{Max}\ A$$

of locales in $\mathbb{B}$ to be constructed, in exactly the same way as that in the topos $\mathbb{E}$ for the seminormed algebra A. In turn, this determines a geometric morphism

$$\mathbb{M}\mathrm{Fn}\ \gamma^*A \to \mathbb{M}\mathrm{ax}\ \gamma^*A$$

between the toposes of sheaves in $\mathbb{B}$ on each of these locales, for which the diagram

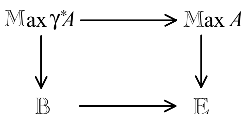


commutes.

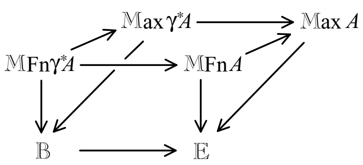
Moreover, the observation that the topos $\mathbb{M}\mathrm{ax}\ \gamma^*A$ is obtained by considering the geometric theory which is the inverse image of that determined by the algebra A in the topos $\mathbb{E}$ means that there is a canonical geometric morphism

$$\mathbb{M}\mathrm{ax}\ \gamma^*A \to \mathbb{M}\mathrm{ax}\ A$$

for which the diagram



is a pullback. An analogous remark concerning the locales of multiplicative linear functionals leads to the consideration of the diagram



in which it may be straightforwardly checked that the top square also commutes.

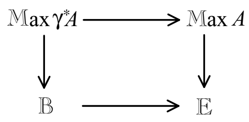
Considering this commutative square on top of the diagram, it will be recalled that our aim is to prove that the canonical morphism

$$\mathbb{M}\mathrm{Fn}\ A \to \mathbb{M}\mathrm{ax}\ A$$

is a covering. Since coverings behave like quotient maps in the category of toposes, it suffices to prove that the composite of the geometric morphisms round the back of that square is a covering. That the geometric morphism

$$\mathbb{M}\mathrm{ax}\ \gamma^*A \to \mathbb{M}\mathrm{ax}\ A$$

is a covering follows from the observation that the diagram



is a pullback, together with the fact that the morphism from $\mathbb{M}\mathrm{ax}\ A$ to $\mathbb{E}$ is determined by a locale $\mathrm{Max}\ A$ in $\mathbb{E}$ which is compact and regular. The pullback of the covering

$$\mathbb{B} \to \mathbb{E}$$

is then a covering.

It remains therefore to show that the canonical morphism

$$\mathbb{M}\mathrm{Fn}\ \gamma^*A \to \mathbb{M}\mathrm{ax}\ \gamma^*A$$

is also a covering, which will be done by showing that the canonical map

$$\mathrm{MFn}\ \gamma^*A \to \mathrm{Max}\ \gamma^*A$$

of locales in $\mathbb{B}$ which determines it is a quotient map, that is, has its inverse image mapping injective. The assertion which it is required to prove concerning the algebra A in the topos $\mathbb{E}$ has therefore been pulled back to the analogous assertion concerning the inverse image algebra γ^*A in the topos $\mathbb{B}$, in which (AC) is satisfied.

Before proving this assertion in $\mathbb{B}$, observe that were one able to remark that γ^*A was again a commutative C*-algebra, the proof would now be quickly completed: for, the canonical map

$$\mathrm{MFn}\ \gamma^*A \to \mathrm{Max}\ \gamma^*A$$

would then be between compact, completely regular locale in a topos satisfying (AC). Hence, by our corollaries in the preceding sections, it would be exactly the continuous mapping from the spectrum of the C*-algebra γ^*A to its maximal spectrum, given by assigning to each functional its kernel. Since this mapping is a homeomorphism, by the Gelfand-Mazur theorem applied in this classical context, it would be, in particular, a quotient map, establishing the required result.

However, the inverse image γ^*A of the commutative C*-algebra A along the Barr covering

$$\gamma :\ \mathbb{B} \to \mathbb{E}$$

is not necessarily a commutative C*-algebra in the topos $\mathbb{B}$. That this is so even when considering a Barr covering of a topos $\mathbb{E}$ may be confirmed by examining the case of the topos of sheaves on a topological space X, of which a Barr covering is given by the Godement covering (Godement [7]), induced by the continuous mapping into X from its discretisation X_{dis}. In this case, the inverse image of a C*-algebra in $\mathrm{Sh}\,X$ is the sheaf on X_{dis} obtained by taking the stalk of the algebra at each $x \in X$. In particular, the inverse image of the C*-algebra of continuous complex functions on X is that given by taking over each $x \in X$ the algebra of germs of continuous complex functions at that point, which generally is neither normed nor complete.

Although the inverse image $\gamma^* A$ of the commutative C*-algebra A along the Barr covering

$$\gamma : \mathbb{B} \to \mathbb{E}$$

of the topos $\mathbb{E}$ may not, therefore, be assumed to be a commutative C*-algebra, it has already been remarked that it satisfies the geometric conditions defining a commutative pre-C*-algebra in the topos $\mathbb{B}$. Considering the canonical map

$$\text{MFn } \gamma^* A \to \text{ Max } \gamma^* A \ ,$$

one remarks immediately that the locale MFn $\gamma^* A$ is compact, merely by the fact that $\gamma^* A$ is a seminormed algebra in $\mathbb{B}$, while the complete regularity of the locale Max $\gamma^* A$ follows by the remark made earlier that its proof for the maximal spectrum depends only on the conditions defining a commutative pre-C*-algebra in the topos. Now, the compactness of MFn $\gamma^* A$ and the regularity of Max $\gamma^* A$ imply that to show that the canonical map is a quotient map, it is enough to prove that it is dense, by the conventional wisdom that the image of a compact locale is compact, hence closed if in a regular locale. Here, a map of locales is said to be dense provided that its inverse image mapping reflects the bottom element of the lattice. Indeed, it is enough to check that if the inverse image of any element of a basis of Max $\gamma^* A$ is the zero of MFn $\gamma^* A$, then it is already the zero of Max $\gamma^* A$.

However, the fact that the proof of complete regularity for Max $\gamma^* A$ depends only on the fact that $\gamma^* A$ is a commutative pre-C*-algebra entails that those propositions of the theory $\mathbb{M}$ax $\gamma^* A$. Recalling that the order relation in these locales is given by provable entailment in the corresponding theory, it may be observed that the denseness of the map is equivalent to the assertion that

$$a \in P \vdash \mathit{false} \text{ in } \mathbb{M}\text{Fn } \gamma^* A \quad \text{implies} \quad a \in P \vdash \mathit{false} \text{ in } \mathbb{M}\text{ax } \gamma^* A \ ,$$

which will now be established. For suppose that $a \in P \vdash \mathit{false}$ in the theory $\mathbb{M}$Fn $\gamma^* A$. Denoting by $\tilde{a} \in B$ the image of $a \in \gamma^* A$ in the completion

$$\gamma^* A \to B \ ,$$

the functoriality of the construction of the spectrum implies that $\tilde{a} \in P \vdash \mathit{false}$ in the theory $\mathbb{M}$Fn B. However, $\mathbb{M}$Fn B is the theory defining the spectrum of a commutative C*-algebra in a topos satisfying (AC) : hence, it is exactly the space of multiplicative linear functionals on B, in which the observation that the open subset $\tilde{a} \in P$ is empty implies that the element $\tilde{a} \in B$ is zero, by the fact that the multiplicative linear functionals separate the elements of B. But then, it follows that the element $a \in \gamma^* A$ has seminorm zero in the seminormed algebra $\gamma^* A$, formally noted by saying that $a \in N^*(q)$ for each positive rational q. Transferring our attention to the theory $\mathbb{M}$ax $\gamma^* A$, this entails that

$$a \in A(q) \vdash \mathit{false} \quad \text{in the theory } \mathbb{M}\text{ax } \gamma^* A \ ,$$

for each positive rational q, by axiom (A2) of this theory. Hence,

$$a \in P \vdash \mathit{false} \ ,$$

by axiom (A8) , which proves that the canonical map is dense, hence a quotient. It may be remarked at this point that it is necessary here to have observed that the propositions of the form $a \in P$ form a base for the theory of maximal ideals of a commutative pre-C*-algebra, in order for this argument to go through. Taken with our previous observations, this establishes in turn that the canonical map

$$\text{MFn } A \to \text{ Max } A$$

of locales in the topos $\mathbb{E}$ is indeed a quotient map, which completes the first half of the proof.

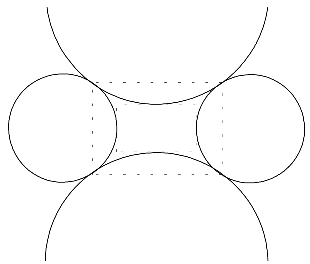
Before going on to prove that this map is also an embedding of locales in the topos $\mathbb{E}$, that is, that its inverse image mapping is surjective, it will be helpful to make some observations about the topology of the complex plane, motivated by the form of the inverse image mapping with which we shall be concerned. This assigns to each proposition $a \in A(q)$ of the theory $\mathbb{M}$ax A the proposition of $\mathbb{M}$Fn A which our convention also allows us to denote by $a \in A(q)$. It will be recalled that this is defined to be the disjunction $\bigvee a \in (r,s)$, taken over all open intervals (r,s) which are rather below the open codisc $A(q)$ of radius q about the origin in the locale of complex numbers.

Thus, while in $\mathbb{M}$Fn A the propositions introduced focus our attention on what is a natural basis for the complex numbers, namely the rational open rectangles (r,s) in terms of which the locale of complex numbers is defined, in considering $\mathbb{M}$ax A one is concerned instead with propositions that reflect the importance of the open codiscs $A(q)$ of radius q about the origin of the complex plane. Of course, if these open codiscs are translated around within the complex plane, considering for each rational complex number λ the open codisc $A(\lambda, q)$ of radius q about λ, which can indeed be straightforwardly defined within the locale of complex numbers, one arrives at an alternative, albeit rather unlikely, subbasis for any bounded region of the complex plane. It will be seen shortly why only bounded regions need be considered in this context.

More explicitly, any rational open rectangle (r,s) in any bounded region of the complex plane may be expressed in the form

$$\bigvee A(\lambda_1^q, r_1^q) \wedge \cdots \wedge A(\lambda_4^q, r_4^q)$$

of a join, taken over all sufficiently small positive rationals q, of the meet of the codiscs which are the exteriors of the four circles each of which passes through the endpoints of one side of the rectangle and through the midpoint of the corresponding side of the rectangle obtained by shrinking in a distance q from each side:



The constraint on the smallness of the positive rational q is only that needed to ensure that the intersection of these codiscs, relative to the bounded region of the complex plane under consideration, lies entirely within the rational open rectangle (r, s) .

A straightforward calculation shows that the centres $\lambda_1^q, \ldots, \lambda_4^q$ and the radii $r_1^q, \ldots, r_4^q$ of these circles are indeed rational, provided that the rectangle (r, s) is rational, for any rational q . Moreover, the fact that the rectangle (r, s) is exactly the join of the concave rectangles obtained in this way, within the bounded region of the plane considered, for each rational $q > 0$ is simply the observation that each of these concave rectangles contains the smaller rectangle, obtained by shrinking each side of (r, s) in by an amount q , which it surrounds, and of which the rectangle (r, s) is the join.

Now, consider again the canonical map

$$\text{MFn}\, A \rightarrow \text{Max}\, A$$

of locales. It must be proved that the inverse image mapping is surjective, for which it suffices to show that for each of the propositions $a \in (r, s)$ which generate the theory of MFn A there exists a proposition of $\mathbb{M}\text{ax}\, A$ which is assigned to it by the inverse image mapping. The above discussion provides a ready candidate for such a proposition. For, choosing a positive rational t for which $a \in A$ lies in the open ball of radius t of the C*-algebra A , one may express the rational open rectangle (r, s) , relative to the bounded region $N(t)$ of the complex plane, in the form

$$\bigvee A(\lambda_1^q, r_1^q) \wedge \cdots \wedge A(\lambda_4^q, r_4^q) ,$$

provided that q is taken sufficiently small with respect to this region, by our remarks above. By earlier observations concerning the propositions of $\mathbb{M}\text{Fn}\, A$ made when our conventions in respect of these propositions were introduced, it follows that $a \in (r, s)$ is provably equivalent to the proposition

$$\bigvee a \in A(\lambda_1^q, r_1^q) \wedge \cdots \wedge a \in A(\lambda_4^q, r_4^q) ,$$

corresponding to this disjunction in the locale of complex numbers. All that remains is to show that for any $a \in A$, any rational complex number λ , and any positive rational q , there is a proposition of $\mathbb{M}\text{ax}\, A$ which has inverse image the proposition $a \in A(\lambda, q)$ of $\mathbb{M}\text{Fn}\, A$.

Our assertion is that the proposition

$$a - \lambda.1 \in A(q)$$

of the theory $\mathbb{M}\text{ax}\, A$ has this property. For, this proposition has inverse image the proposition $a - \lambda.1 \in A(q)$ of $\mathbb{M}\text{Fn}\, A$, by definition of the inverse image mapping. Further, this proposition of $\mathbb{M}\text{Fn}\, A$ is defined, by our conventions, to be the disjunction

$$\bigvee a - \lambda.1 \in (r, s)$$

taken over all rational open rectangles (r, s) which are rather below $A(q)$ in the locale of complex numbers. But, the rational open rectangle (r, s) is rather below $A(q)$ exactly if the open rectangle $(\lambda + r, \lambda + s)$ is rather below $A(\lambda, q)$. Moreover, the proposition $a - \lambda.1 \in (r, s)$ is provably equivalent in $\mathbb{M}\text{Fn}\, A$ to the proposition $a \in (\lambda + r, \lambda + s)$.

Hence, the disjunction is equivalent in $\mathbb{M}\text{Fn}\, A$ to that defining the proposition $a \in A(\lambda, q)$, as required.

Coupling this observation with that made in the preceding paragraph concerning any proposition of $\mathbb{M}\text{Fn}\, A$ of the form

$$a \in (r, s) ,$$

one sees that there is indeed a proposition of the theory $\mathbb{M}\text{ax}\, A$ of which it is the inverse image under the canonical map

$$\text{MFn}\, A \rightarrow \text{Max}\, A .$$

It follows that the inverse image mapping is surjective, in addition to being injective, which completes the proof of the Gelfand-Mazur theorem.

It will be recalled that this map

$$\text{MFn}\, A \rightarrow \text{Max}\, A$$

of locales which the Gelfand-Mazur theorem has now shown to be an isomorphism is to be considered intuitively to be that which assigns to each multiplicative linear functional on the commutative C*-algebra A the maximal ideal which is its kernel. It must be understood that this statement needs to be interpreted constructively if it is to have any explicit validity. Thus, one is considering in fact not the maximal ideal which is the kernel, but rather the prime of the commutative C*-algebra A which is its constructive counterpart. In particular, one is considering not those elements of A which are mapped to zero by the functional, but rather these which yield complex numbers which are bounded away from zero with respect to the coseminorm. However, it may be recalled that, in the theory of the locale Max A , it is provable that the propositions

$$a \in A(q)$$

which determine this coseminorm are expressible in terms of the propositions

$$a \in P$$

which determine the prime. Explicitly, the proposition $a \in A(q)$ is provably equivalent to the proposition $(a - q.1)^+ \in P$, in which the element $(a - q.1)^+ \in A$ may be obtained algebraically from the element $a \in A$. Hence, on the one hand, the concept of maximal ideal in this constructive sense is indeed still one which is describable algebraically, while, on the other hand, the prime concerned is all that is needed to know the coseminorm also. It is simply that the most convenient way of describing the concept involved is by bringing in the seminorm of the commutative C*-algebra A . That the theory can be described algebraically, however, indicates that the seminorm of the commutative C*-algebra A may also be recovered from its algebraic structure, as is the case classically.

The Gelfand-Mazur theorem then states constructively that each maximal ideal of the commutative C*-algebra is the kernel of a unique multiplicative linear functional. In particular, one has the following:

Corollary. For any commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$ in which the Axiom of Choice is satisfied, the Gelfand-Mazur isomorphism establishes a homeomorphism from the space of multiplicative linear functionals on A to the space of maximal ideals of A.

Since the logic of the topos $\mathbb{E}$ is classical, in the presence of the Axiom of Choice, the concept of maximal ideal to which this refers is then exactly that taken classically. For the locale $\text{Max}\, A$ then has points given by the closed prime ideals of A, which for a commutative C*-algebra are exactly the maximal ideals of A.

Finally, it may be remarked that, from the Gelfand duality which may be established [4] based on the form of the Gelfand-Mazur theorem proved here, it may be deduced that in any Grothendieck topos $\mathbb{E}$ any commutative C*-algebra A that is a local ring is necessarily isometrically *-isomorphic to the algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in the topos $\mathbb{E}$. The proof of this, along with other consequences of Gelfand duality in a Grothendieck topos, will be left to the successor of this paper [4].

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