A Geometric Characterization concerning Compact, Convex Sets

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In this paper, we are concerned with establishing a characterization of any compact, convex set K in a normed space A in an arbitrary topos $\mathbb{E}$ with natural number object. The characterization is geometric, not in the sense of categorical logic, but in the intuitive one, of describing any compact, convex set K in terms of simpler sets in the normed space A. It is a characterization of the compact, convex set in the sense that it provides a necessary and sufficient condition for an element of the normed space to lie within it. Having said this, we should immediately qualify our statement by stressing that this is the intuitive content of what is proved; the formal statement of the characterization is required to be in terms appropriate to the constructive context of the techniques used.

The idea involved in formulating the characterization in a way which may be interpreted constructively is quite straightforward. In various classical results on convex sets, notably the geometric Hahn-Banach theorem and the Krein-Milman theorem, use is made, sometimes only implicitly, of pairs of hyperplanes which bound a compact, convex set K. The hyperplanes concerned are obtained by taking a linear functional φ, which may be assumed to have norm not exceeding one on the normed space A, together with an open interval (r,s), which may be taken to have rational end-points, on the real line $\mathbb{R}$, satisfying the condition that

$$K \subseteq \varphi^{-1}(r,s).$$

Calling a linear functional φ together with a rational open interval (r,s) satisfying this condition a bound on the compact, convex set K, the characterization of the set K that may be proved classically is as follows: the points of compact, convex set K are those that are bounded by every bound of K. More geometrically, a compact, convex set K is the intersection of those open regions bounded by its bounding hyperplanes. .

It is in this form that the characterization of any compact, convex set in a normed space A may be proved constructively, based on a form of the Hahn-Banach theorem which has been established in complete constructive generality. The important point in the formulation of the characterization is that it may be stated in terms of a property of linear functionals on the normed space A, namely that of bounding the compact, convex set K in relation to a particular rational open interval (r,s) on the real line. In turn, this allows the concept of an element $x \in A$ of the normed space being bounded by every bound of K to be expressed in terms of the propositional theory of linear functionals on A of norm not exceeding one, which is the

context within which the Hahn-Banach theorem is formulated constructively.

As a consequence of the characterization established, a constructive form of the geometric Hahn-Banach theorem may be obtained, in terms of a statement that a point of the normed space A lies in the closed convex hull of a given bounded subset of A precisely if it is bounded by every bound of the subset. It is interesting to remark that classically the bounds of a compact, convex set and of its set of extreme points are identical. It is the absence of any constructive counterpart of the existence of sufficiently many extreme points of any compact, convex set which is then the obstruction to obtaining a constructive formulation of the Krein-Milman theorem.

1. Convexity and well-boundedness

The context within which the results will be established is that of a normed linear space A over the field $\mathbb{R}$ of real numbers in a topos $\mathbb{E}$, assumed therefore to have a natural number object $\mathbb{N}_{\mathbb{E}}$. The conventions adopted in working within this context will be those which may be found in [3], in which the Hahn-Banach theorem valid in any Grothendieck topos $\mathbb{E}$ was established by the present authors. The theorem was subsequently shown by Vermeulen [5] to be valid in any topos $\mathbb{E}$ having a natural number object, thereby establishing the theorem constructively. For further discussion of the concept of a normed linear space A in a topos $\mathbb{E}$, together with the ideas associated with that, reference may be made to the paper of Burden and Mulvey [1], where, together with that [3] of the present authors, indication may also be found of the motivation for considering these results within the constructive context of an arbitrary topos.

Definition. A subset K of a normed linear space A will be said to be *compact* provided that any open covering of K has a finite subcovering.

It should be noted that the concept of finiteness involved in this definition is that of Kuratowski finiteness [2] in the topos, which is equivalent to the existence of an indexing of the subcovering by a surjective mapping from the subset

$$\{ i \in \mathbb{N} \mid i < n \}$$

of the set of natural numbers $\mathbb{N}_{\mathbb{E}}$ determined by some element $n \in \mathbb{N}_{\mathbb{E}}$.

It may be proved that any compact subset K of a normed linear space A is closed, in the sense that any element $x \in A$ to which converges a Cauchy approximation

$$C : \mathbb{N}_{\mathbb{E}} \to \Omega^A$$

contained in K satisfies $x \in K$. A compact subset K is also bounded, in the sense that there exists an open ball $N(q)$, determined by the norm

$$N : \mathbb{Q}^+_{\mathbb{E}} \to \Omega^A$$

on the linear space A, which contains the subset K. It may be remarked that any bounded linear functional maps bounded sets to bounded sets, by its boundedness.

Definition. A subset K of a normed linear space A will be said to be *convex* provided that

$$x, y \in K \text{ implies } \alpha x + (1-\alpha) y \in K$$

for any $a \in \mathbb{R}_{\mathbb{E}}$ satisfying $0 \le a \le 1$.

Of course, it follows that a convex subset K of A is also closed under convex linear combinations

$$\sum a_i x_i$$

indexed by finite sets. It may be remarked that any bounded linear functional maps convex sets to convex sets, by its linearity.

It is known that, in general, a normed linear space A in a topos $\mathbb{E}$ need not have adequately many bounded linear functionals. In particular, the linear space of bounded linear functionals on A may be the zero space, even though the space A is not itself zero: see [3]. Moreover, bounded linear functionals defined on a subspace A' of a normed linear space A may not admit extensions to the space A itself.

The Hahn-Banach theorem which applies to a seminormed space A in a topos $\mathbb{E}$ is therefore stated in terms of the locale

$$\mathrm{Fn}\, A$$

of linear functionals on the space A of norm not exceeding one. This dual locale is obtained by describing its generators and relations in terms of propositions

$$a \in (r, s)$$

for each $a \in A$ and each rational open interval (r, s), which intuitively assert of a linear functional that it maps $a \in A$ into the open interval (r, s). The relations between these generators express the axioms which relate these propositions in the theory $\mathbb{F}\mathrm{n}\, A$ of a linear functional on A of norm not exceeding one.

Equivalently, however, the generator

$$a \in (r, s)$$

may be considered to represent the subbasic open subset of the weak* topology on the unit ball of the dual of the seminormed space A, consisting of all functionals which map $a \in A$ into the rational open interval (r, s). The axioms of the theory $\mathbb{F}\mathrm{n}\, A$ now interpret as relations between these subbasic open subsets of the weak* topology.

The Hahn-Banach theorem is the assertion that, for any subspace A' of a seminormed space A in the topos $\mathbb{E}$, the canonical map

$$\mathrm{Fn}\, A \to \mathrm{Fn}\, A'$$

induced by the inclusion of A' in the space A is a quotient map of compact, completely regular locales. In the classical situation, this mapping is that between the locales of open subsets of the unit balls of the dual spaces, induced by restriction of linear functionals, thereby recovering the Hahn-Banach theorem in its more usual form. Constructively, however, the theorem stated in terms of dual locales provides the consequences usually associated with the Hahn-Banach theorem, yielding in particular the isometric embedding of any normed linear space A in the space $\mathbb{R}(\mathrm{Fn}\, A)$ of continuous real functions on the dual

locale, which will later be needed.

For the moment, we note that the observation that linear functionals on a normed linear space A need not exist in any abundance means that the intuitive concept of a bound on a compact, convex subset K may not itself be of great value. However, the notion of an element $x \in A$ being bounded by every bound of the subset K can be stated immediately in terms of open subsets of the weak* topology on the dual of A, thereby allowing the following definition to be made:

Definition. An element $x \in A$ of the normed linear space A in the topos $\mathbb{E}$ will be said to be *well-bounded* by a subset K of A provided that the entailment

$$\bigwedge_{a \in K} a \in (r, s) \vdash x \in (r, s)$$

is provable in the theory $\mathbb{F}\mathrm{n}\, A$ for each rational open interval (r, s).

Classically, the condition expresses exactly that, for each rational open interval (r, s), every functional which maps $a \in K$ into (r, s) for each $a \in K$ also maps $x \in A$ into (r, s). In other words, each bound of the subset K is a bound of $x \in A$.

With these preliminaries, we may now state the theorem characterizing any compact, convex subset K of a normed linear space A in the topos $\mathbb{E}$.

Theorem 1.1. *For any compact, convex subset K of a normed linear space A in the topos $\mathbb{E}$, the following assertions concerning an element $x \in A$ are equivalent:*

(i) $x \in K$ *;*

(ii) $x \in A$ *is well-bounded by the subset K.*

The theorem, of which the proof will occupy most of the rest of the paper, asserts that a compact, convex subset of A coincides, intuitively, with its bounded hull in A. In particular, it will be seen later that from this can be deduced a geometric form of the Hahn-Banach theorem relating to the closed convex hulls of subsets of the normed linear space A.

2. Canonical embedding

The idea behind the proof of the theorem is extremely simple. The dual locale Fn A of the normed linear space A in the topos $\mathbb{E}$ determines a topos $\mathbb{F}\mathrm{n}\, A$ defined over $\mathbb{E}$, in the sense of the existence of a canonical map

$$\mathbb{E} \leftarrow \mathbb{F}\mathrm{n}\, A$$

of toposes. The topos $\mathbb{F}\mathrm{n}\, A$, realized by taking the category of sheaves on the dual locale Fn A in the topos $\mathbb{E}$, is the classifying topos of the theory of linear functionals on A of norm not exceeding one, in the sense, described more fully below, that there is a linear functional

$$A \to \mathbb{R}_{\mathrm{Fn}\, A}$$

from the normed linear space A in the topos $\mathbb{E}$ to the space $\mathbb{R}_{\mathrm{Fn}\, A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\, A$.

Explicitly, the linear functional, defined as a map in the topos $\mathbb{F}\mathrm{n}\, A$ from the inverse image of the space A along the map of toposes

$$\mathbb{F}\mathrm{n}\, A \to \mathbb{E}$$

to the space $\mathbb{R}_{\mathrm{Fn}\, A}$ assigns to the inverse image of each $a \in A$ the continuous real function on the locale Fn A of which the inverse image of each rational open interval (r, s) is the element of the locale Fn A given by the proposition

$$a \in (r, s).$$

The linear functional thus defined is generic in the sense that any linear functional $A \to \mathbb{R}_{\mathbb{E}}$ of norm not exceeding one in any topos $\mathbb{F}$ defined over $\mathbb{E}$ is obtained by taking the inverse image of the linear functional along a uniquely determined map

$$\mathbb{F} \to \mathbb{F}\mathrm{n}\, A$$

of toposes defined over $\mathbb{E}$.

Intuitively, considering the locale Fn A to be the constructive equivalent of the space of linear functionals on A of norm not exceeding one, the space $\mathbb{R}_{\mathrm{Fn}\, A}$ of real numbers in the topos of sheaves on the locale Fn A is equivalent to the normed linear space over Fn A of which each fibre is the space of real numbers $\mathbb{R}_{\mathbb{E}}$ in the topos $\mathbb{E}$. The linear functional

$$A \to \mathbb{R}_{\mathrm{Fn}\, A}$$

is that which maps the space A into the space of real numbers at each point of the unit ball of the dual of A by the linear functional on A given by that point of the unit ball.

The observation, classically coming from the Hahn-Banach theorem, that the linear functionals in the unit ball of the dual of A are jointly isometric is reflected constructively in the fact that the linear functional

$$A \to \mathbb{R}_{\mathrm{Fn}\, A},$$

now considered adjointly to be the map

$$A \to \mathbb{R}(\mathrm{Fn}\, A)$$

in the topos $\mathbb{E}$ from the normed linear space A to the space of continuous real functions on the locale Fn A, is isometric. Explicitly, this assigns to each $a \in A$ the map

$$\hat{a} : \mathrm{Fn}\, A \to \mathbb{R}_{\mathbb{E}}$$

of locales in $\mathbb{E}$, of which the inverse image is given by the assignment $(r, s) \mapsto a \in (r, s)$ defined above.

Consider now a compact, convex subset K of the normed linear space A in the topos $\mathbb{E}$. Concerning the image $K_{\mathbb{R}}$ of the subset K along the linear functional

$$A \to \mathbb{R}_{\mathrm{Fn}\, A}$$

one has the following result:

Proposition 2.1. *The canonical image $K_{\mathbb{R}}$ of a compact, convex subset K of a normed linear space A in the topos $\mathbb{E}$ along the linear functional*

$$A \to \mathbb{R}_{\mathrm{Fn}\, A}$$

in the topos $\mathbb{F}\mathrm{n}\, A$ is a compact, convex subset of the space $\mathbb{R}_{\mathrm{Fn}\, A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\, A$.

For, denoting by

$$\beta : \mathbb{F}\mathrm{n}\, A \to \mathbb{E}$$

the map of toposes from $\mathbb{F}\mathrm{n}\, A$ to $\mathbb{E}$, the inverse image functor assigns to the normed space A in the topos $\mathbb{E}$ the constant sheaf $\beta^* A$ in the topos $\mathbb{F}\mathrm{n}\, A$, together with the seminorm of which the open ball of radius q is the inverse image $\beta^* N(q)$ of that of A, for each $q \in \mathbb{Q}_{\mathbb{E}}^{+}$. The inverse image $\beta^* K$ of the compact, convex subset K of the normed linear space A is therefore compact, since its topology is induced by that of A and the locale Fn A is compact, and convex, since its algebraic structure is induced by that of A.

Since $K_{\mathbb{R}}$ is the image of $\beta^* K$ along the mapping

$$\beta^* A \to \mathbb{R}_{\mathrm{Fn}\, A},$$

the continuity and linearity of the latter ensure that the compactness and convexity of $K_{\mathbb{R}}$ follow from that of the inverse image $\beta^* K$ in the seminormed space $\beta^* A$.

3. Closed bounded intervals

By considering the canonical linear functional $A \to \mathbb{R}_{\mathrm{Fn}\, A}$ along the map $\mathbb{E} \leftarrow \mathbb{F}\mathrm{n}\, A$ of toposes, the compact, convex subset K of the normed linear space A in the topos $\mathbb{E}$ has been mapped to the compact, convex subset $K_{\mathbb{R}}$ of the normed space $\mathbb{R}_{\mathrm{Fn}\, A}$ in the topos $\mathbb{F}\mathrm{n}\, A$. Of course, classically, the compact, convex subsets of the space $\mathbb{R}$ of real numbers are exactly the closed bounded intervals, motivating the following

Definition. A closed subset I of the space $\mathbb{R}_{\mathbb{F}}$ of real numbers in a topos $\mathbb{F}$ will be said to be a *closed bounded interval* of $\mathbb{R}_{\mathbb{F}}$ provided that

(i) the subset I is contained in some rational open interval (r, s) of $\mathbb{R}_{\mathbb{F}}$; and

(ii) the subset I is the intersection of those rational open intervals (r, s) of $\mathbb{R}_{\mathbb{F}}$ that contain it.

With these preliminaries, one may now prove the following result:

Proposition 3.1. *In the space $\mathbb{R}_{\mathbb{F}}$ of real numbers in the topos $\mathbb{F}$, any compact, convex set I is a closed bounded interval.*

Suppose that I is a compact, convex subset of $\mathbb{R}_{\mathbb{F}}$. Then necessarily I is a closed, bounded subset of $\mathbb{R}_{\mathbb{F}}$ since it is compact. That it is a closed interval is shown as follows. Suppose that $x \in \mathbb{R}_{\mathbb{F}}$ lies in each rational open interval (r, s) that contains the subset I. Then it is asserted that $x \in I$. For consider, for each $n \in \mathbb{N}_{\mathbb{F}}$ the subset

$$C_n = \{ y \in I \mid |x - y| < 1/n \}.$$

Then, by the closedness of the subset I, it is enough to show that for each $n \in \mathbb{N}_{\mathbb{F}}$ there exists $y \in \mathbb{R}_{\mathbb{F}}$ such that $y \in C_n$.

Given $n \in \mathbb{N}_{\mathbb{F}}$ there exists an open covering of the compact subset I by open neighbourhoods

$$U_i = \{ y \in \mathbb{R}_{\mathbb{F}} \mid |y - y_i| < 1/n \}$$

for finitely many $y_1, \ldots, y_m \in I$. Then, letting U denote the open complement of the closed subset I, the open subsets $U_1, \ldots, U_m$, together with U, cover the space $\mathbb{R}_{\mathbb{F}}$. Hence, $x \in U_i$ for some i, or $x \in U$.

Now, if $x \in U$, then there exists a rational open interval (p, q) such that

$$x \in (p, q) \subseteq [p, q] \subseteq U.$$

One therefore concludes that the subset I is contained in the union of the open intervals $(-\infty, p), (q, \infty)$. But I convex implies that $I \subseteq (-\infty, p)$ or $I \subseteq (q, \infty)$, provided that I is inhabited. Then by the boundedness of I, it follows that the rational open interval concerned may be taken to be bounded, hence that there exists a rational open interval (r, s) containing the subset I, yet disjoint from the open interval (p, q). But, by hypothesis, $I \subseteq (r, s)$ implies $x \in (r, s)$, contradicting the fact that $x \in (p, q)$. Hence $x \in U$ is false.

Recalling that the open subsets $U_1, \ldots, U_m$, together with U, cover the space $\mathbb{R}_{\mathbb{F}}$, one concludes that $x \in U_i$ for some i. Choosing such an i, one then has that

$$|x - y_i| < 1/n,$$

by the definition of U_i, hence that

$$y_i \in C_n.$$

Each of the subsets C_n of I is therefore inhabited, showing that C is a Cauchy approximation on I, evidently converging to $x \in \mathbb{R}_{\mathbb{F}}$. Thus $x \in I$, as required, by the closedness of the compact subset I. Hence I is a closed bounded interval of $\mathbb{R}_{\mathbb{F}}$.

The condition required of a closed subset I of $\mathbb{R}_{\mathbb{F}}$ for it to be a closed interval may, of course, be restated in the following form: that for any $x \in \mathbb{R}_{\mathbb{F}}$ one has that $x \in I$ if and only if $x \in \mathbb{R}_{\mathbb{F}}$ is bounded by each bound of I, in the evident sense in terms of rational open intervals which contain it.

The relationship of this condition with that concerning well-boundedness considered earlier may be made explicit in the case that the compact, convex subset is the subset $K_{\mathbb{R}}$ of the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$, determined by a compact, convex subset K of a normed linear space A in the topos $\mathbb{E}$. In that case, one has the following result:

Corollary 3.2. *For any compact, convex subset K of a normed linear space A in a topos $\mathbb{E}$, the following assertions concerning an element $x \in A$ are equivalent:*

(i) *the element $x \in A$ is well-bounded by K in the topos $\mathbb{E}$;*

(ii) *the canonical image $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ of the element $x \in A$ lies in the subset $K_{\mathbb{R}}$ of the space of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$.*

Suppose first that $x \in A$ is well-bounded by K, hence that

$$\bigwedge_{a \in K} a \in (r, s) \vdash x \in (r, s)$$

is provable in the theory $\mathbb{F}\mathrm{n}\,A$ for any rational open interval (r, s). Then, observing that the locale Fn A consists of the propositions of the theory $\mathbb{F}\mathrm{n}\,A$, ordered by provable entailment, modulo provable equivalence in the theory $\mathbb{F}\mathrm{n}\,A$, one may conclude that the meet of the elements

$$a \in (r, s)$$

taken over all $a \in K$ is contained in the element

$$x \in (r, s)$$

of the locale Fn A, for any rational open interval (r, s) in the topos $\mathbb{E}$.

Noting that any rational number in the topos $\mathbb{F}\mathrm{n}\,A$ is locally the inverse image of a rational number in the topos $\mathbb{E}$, and recalling from [3] that the element $y \in (r, s)$ of the locale Fn A is the extent to which the canonical image $\hat{y} \in \mathbb{R}_{\mathrm{Fn}\,A}$ lies in the rational open interval (r, s), for any $y \in A$, one sees that $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ is bounded by each bound of the canonical image $K_{\mathbb{R}}$ of the compact, convex subset K along the canonical linear functional

$$A \to \mathbb{R}_{\mathrm{Fn}\,A}.$$

But $K_{\mathbb{R}}$ being compact and convex in the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$ implies that $K_{\mathbb{R}}$ is a closed bounded interval, by the preceding proposition. Hence, since the element $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ satisfies the condition

$$K_{\mathbb{R}} \subseteq (r, s) \quad \text{implies} \quad \hat{x} \in (r, s)$$

for each rational open interval (r, s), one may conclude that $\hat{x} \in K_{\mathbb{R}}$. Therefore, $x \in A$ is well-bounded by K in the topos $\mathbb{E}$ implies that $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ is bounded by each bound of the closed interval $K_{\mathbb{R}}$ in the topos $\mathbb{F}\mathrm{n}\,A$.

The converse of the argument also holds, since $\hat{x} \in K_{\mathbb{R}}$ implies that $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ is bounded by each bound of the closed interval $K_{\mathbb{R}}$ in the topos $\mathbb{F}\mathrm{n}\,A$. Hence, the meet

$$\bigwedge_{a \in K} a \in (r, s)$$

is contained in the element $x \in (r, s)$ of the locale Fn A, for each rational open interval (r, s). By the construction of the locale Fn A, one has that

$$\bigwedge_{a \in K} a \in (r, s) \vdash x \in (r, s)$$

is provable in the theory $\mathbb{F}\mathrm{n}\,A$. Thus, $x \in A$ is well-bounded by the compact, convex subset K of the normed linear space A.

4. Compact, convex sets

The idea so far has been to map the normed linear space A in the topos $\mathbb{E}$ isometrically into the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$ of sheaves on its dual locale. In so doing, the compact, convex subset K of the normed linear space A has been mapped to a compact, convex subset $K_{\mathbb{R}}$ of the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers of the topos $\mathbb{F}\mathrm{n}\,A$, which is therefore seen to be a closed bounded interval of $\mathbb{F}\mathrm{n}\,A$. Moreover, for any element $x \in A$, the condition of being well-bounded by the compact, convex subset K has been shown to be equivalent to the condition that $\hat{x} \in K_{\mathbb{R}}$ in the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$.

The proof of the theorem asserted, namely that $x \in A$ is well-bounded by the compact, convex subset K if and only if $x \in K$, is therefore completed by establishing the following result:

PROPOSITION 4.1. *For any element $x \in A$ of the normed linear space A in the topos $\mathbb{E}$, the following assertions are equivalent:*

(i) *the element $x \in A$ lies in the compact, convex subset K of the normed linear space A in the topos $\mathbb{E}$;*

(ii) *the element $\hat{x} \in \mathbb{R}_{\mathrm{Fn}\,A}$ lies in the compact, convex subset $K_{\mathbb{R}}$ of the space $\mathbb{R}_{\mathrm{Fn}\,A}$ of real numbers in the topos $\mathbb{F}\mathrm{n}\,A$.*

The equivalence is proved by showing that $\hat{x} \in K_{\mathbb{R}}$ implies $x \in K$, the converse implication being trivially true. Suppose then that $\hat{x} \in K_{\mathbb{R}}$, and recall that $K_{\mathbb{R}}$ is the canonical image of K along the canonical functional $A \to \mathbb{R}_{\mathrm{Fn}\,A}$. Because the elements of $\mathbb{R}_{\mathrm{Fn}\,A}$ which lie in the subset $K_{\mathbb{R}}$ are exactly those which are locally equal to images of elements of A which lie in the subset K, it follows by the compactness of the locale Fn A that there is an open covering of Fn A by finitely many $U_1, \ldots, U_m$ together with, for each i, an element $x_i \in K$ for which

$$\hat{x}\,|U_i = \hat{x}_i\,|U_i.$$

It is asserted that this implies that $x \in A$ lies in the closed convex hull of the elements $x_1, \ldots, x_m \in K$, hence that $x \in K$, since K is already convex.

To establish this, we move into a covering topos $\mathbb{B}$ of the topos $\mathbb{E}$ in which the law of the excluded middle is satisfied. The existence of this Boolean covering

$$\gamma : \mathbb{B} \to \mathbb{E}$$

of the topos $\mathbb{E}$ may be proved constructively (see [2]), so that the conclusion to be reached is constructively valid.

Consider now the subspace A' of the normed linear space A generated by $x \in A$ together with the elements $x_1, \ldots, x_m \in K$. Observe firstly that, by the Hahn-Banach theorem in its constructive form, the canonical map of locales Fn $A \to$ Fn A' induced by the inclusion of the subspace A' in the normed linear space A is a quotient map. Consider the proposition in the theory $\mathbb{F}\mathrm{n}\,A$ of linear functionals of norm not exceeding one on the subspace A' which asserts that a linear functional is positive on $x - x_1, \ldots, x - x_m \in A'$. Then its inverse image in the locale Fn A is provably false, since Fn A is covered by the elements $U_1, \ldots, U_m$ on each of which one of the elements $x - x_1, \ldots, x - x_m \in A'$ is zero, hence cannot be mapped positively. By the Hahn-Banach theorem, it follows that the proposition is provably false in the theory $\mathbb{F}\mathrm{n}\,A$.

Considering the inverse image $\gamma^* A'$ of the subspace A' of the normed linear space A along the Boolean covering introduced above, observe that the proposition which asserts that a linear functional on $\gamma^* A'$ is positive on each of the inverse images $\gamma^*(x - x_1), \ldots,$

$\gamma^*(x - x_m) \in \gamma^* A'$ is provably false in the theory $\mathbb{F}\mathrm{n}\,\gamma^* A'$, having been shown to be so in the theory $\mathbb{F}\mathrm{n}\,A'$.

Denoting by L the convex hull of $x_1, \ldots, x_m \in A'$, consider its inverse image $\gamma^* L$ in the topos $\mathbb{B}$. Observe that, in the Boolean topos $\mathbb{B}$, one may conclude concerning the element $\gamma^* x \in \gamma^* A'$ that either $\gamma^* x \in \gamma^* L$ or not. In the latter case, the classical, finite-dimensional, form of the Hahn-Banach theorem yields the existence of a linear functional

$$\varphi : \gamma^* A' \to \mathbb{R}_{\mathbb{B}}$$

of norm not exceeding one, for which

$$\varphi(\gamma^* x) > \varphi(\gamma^* x_i)$$

for each i, contradicting the observation that a linear functional on $\gamma^* A'$ cannot be positive on all of the inverse images of the elements $x - x_1, \ldots, x - x_m \in A'$. Hence, $\gamma^* x \in \gamma^* L$ must indeed be the case. But then, by the property of a geometric covering of a topos, one may conclude that $x \in L$. Hence $x \in A$ lies in the convex hull of $x_1, \ldots, x_m \in K$. Therefore $x \in K$, since K is already convex, as required.

Hence $x \in K$ if and only if $\hat{x} \in K_{\mathbb{R}}$ if and only if $x \in A$ is well-bounded by K, as required. This completes the proof of the theorem.

5. GEOMETRIC HAHN-BANACH THEOREM

In the form which is often called the geometric Hahn-Banach theorem, one shows classically that, for any element $x \in A$ of a normed linear space A which does not belong to a compact, convex subset K of A, there exists a linear functional of norm not exceeding one for which

$$\varphi(x) > \sup_{y \in K} \varphi(y).$$

Evidently this may be restated contrapositively to yield that an element $x \in A$ of a normed linear space A at which every linear functional of norm not exceeding one satisfies

$$\varphi(x) \le \sup_{y \in K} \varphi(y)$$

must belong to the compact, convex subset K of A.

Of course, in this form the geometric Hahn-Banach theorem may be stated, and proved, constructively as a corollary to the theorem already obtained. More generally, we obtain the following result:

COROLLARY 5.1. *For any bounded subset S of which the closed convex hull in a normed linear space A in a topos $\mathbb{E}$ is compact, the following assertions concerning an element $x \in A$ are equivalent:*

(i) *$x \in A$ is well-bounded by the subset S of the normed linear space A;*

(ii) *$x \in A$ is in the closed convex hull of the subset S in the normed linear space A.*

Consider the closed convex hull L of the bounded subset S. Then $x \in L$ if and only if $x \in A$ is well-bounded by L, by the theorem established above. But $x \in A$ is well-bounded by L if and only if well-bounded by S, since in one direction the implication is clear, whilst, in the other, it is asserted that $x \in A$ is well-bounded by L implies that it is well-bounded by S. For suppose that $x \in A$ is well-bounded by L. Hence

$$\bigwedge_{a \in L} a \in (r,s) \vdash x \in (r,s)$$

is provable in $\mathbb{F}\mathrm{n}\, A$, for any rational open interval (r,s). But

$$\bigwedge_{b \in S} b \in (r,s) \vdash a \in (r,s)$$

for each $a \in L$, since any $a \in L$ may be expressed as the limit of a Cauchy approximation C of which the subsets consist of finite convex linear combinations of elements of S. However $b_1, \ldots, b_m \in S$ and $\beta_1, \ldots, \beta_m \in \mathbb{R}_E$ satisfying $0 \leq \beta_i \leq 1$ and $\Sigma \beta_i = 1$ allow the entailment

$$b_1 \in (r,s) \wedge \cdots \wedge b_m \in (r,s) \vdash b \in (r,s)$$

to be proved in the theory $\mathbb{F}\mathrm{n}\, A$ for

$$b = \Sigma\, \beta_i b_i,$$

by applying the linearity axioms of the theory $\mathbb{F}\mathrm{n}\, A$; while, applying the continuity axioms of the theory, together with a little care, allows one to close up under convergence of Cauchy approximations. Thus $x \in A$ being well-bounded by the closed convex hull L implies that $x \in A$ is well-bounded by the bounded subset S of which the closed convex hull is taken, completing the proof.

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References

1. Burden, C.W., Mulvey, C.J.: Banach spaces in categories of sheaves. *In* Lecture Notes in Mathematics, **753**, 169-196. Springer-Verlag, 1979.

2. Johnstone, P.T.: Topos Theory. Academic Press, 1977.

3. Mulvey, C.J., Wick Pelletier, J.: A globalisation of the Hahn-Banach theorem. Advances in Math., **89** (1991), 1-59.

4. Rudin, W.: Functional Analysis. McGraw-Hill, 1973.

5. Vermeulen, J.J.C.: Constructive techniques in functional analysis. D.Phil. Thesis, University of Sussex, 1986.