On the Geometry of Choice

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Introduction

The Axiom of Choice has, since its conception, formed an important part of the framework within which much of mathematics is done. Not only has its application allowed the proof of results to which it is an apparently essential pre-requisite, but its availability has shaped the way in which the mathematics which it proves has been formulated. There is no need to carry along the way the impedimenta of accumulated detail when selection of an arbitrarily chosen instance can be relied upon in every situation due to the presence of an Axiom of Choice which guarantees its retrieval from a range of possibilities. There is a received wisdom which says that this shaping of mathematics is fine and good, and that the avoidance of intricacy which this encourages leads to a minimality of expression which is characteristic of elegance in mathematics. There is also a reality, perhaps less well received, which says that all may not be quite so fine, quite so good, and that the intricacies avoided in the cause of elegance may be waiting just round the next conceptual corner to pounce on the unsuspecting. Indeed, the joys of life with the Axiom of Choice may actually limit, rather than expand, the extent to which mathematics may be developed.

This paper is about the impact of the Axiom of Choice on one particular part of functional analysis, namely that surrounding the Hahn-Banach theorem. It describes the way in which the Hahn-Banach theorem, which classically is almost, but not quite, equivalent to the Axiom of Choice, may be reformulated in such a way that it may be proved entirely constructively, in particular without application of the Axiom of Choice, while remaining classically equivalent, in the presence of the Axiom of Choice, to its classical formulation. At the centre of this reformulation is the idea that examining the ways in which the theorem fails to remain valid under the change to a context that is propositionally non-classical may itself indicate the path that must be followed towards constructivity. In this sense, the development of mathematical ideas within such contexts achieves a powerful extrinsic value, in addition to its intrinsic merits.

Although the case to be made is specifically aimed at the Hahn-Banach theorem, the ideas behind it may be applied to a wide range of mathematical problems. Indeed, an indication of the extent to which this has already been carried out will be given towards the end of the paper. The temptation, on the basis of the evidence that already exists, is to feel that almost any mathematical result, at least of the kind that is central to the development of the subject and its applications, may with a little care be treated in this way. Moreover, that re-examining mathematics thoughtfully, within a more ascetic world in which use of the Axiom of Choice

is replaced by a constructive approach, may lead to greater insight into what is actually taking place within it.

The form of the paper is largely historical, following the development of the ideas which it examines. With the exception of the final stage of the argument, of which details have yet to be published in a joint paper with J.J.C. Vermeulen [22], the ideas have already been presented in papers by various authors [10,20,24,25]. The presentation here will be kept to the level of detail needed to follow the development of the ideas, the reader being referred to the relevant original sources for a more thorough discussion. The aim instead will be to take the broader view, focusing on the techniques that have been developed and the principles that lie behind. For making those clearer to me, I have to thank principally Bill Lawvere and Japie Vermeulen, whose mathematical insights have that clarity of vision and intensity of perspective that open the eyes of others to new things. Finally, to Bernhard Banaschewski and to Joan Pelletier, as well as to Japie Vermeulen, without whom the work would never have been done and with whom it variously was done, I give warm thanks, both for mathematics and for friendship.

1. The classical Hahn-Banach theorem.

Before considering its constructive aspects, we shall recall the Hahn-Banach theorem in the form with which we shall be concerned, and remind ourselves of its proof. The theorem is straightforwardly stated:

Theorem 1.1. *For any bounded linear functional* φ *on a linear subspace* A *of a seminormed linear space* B *there exists a norm-preserving extension to a linear functional* ψ *defined on the seminormed space* B *itself.*

The theorem is also straightforwardly proved, by applying Zorn's Lemma, which is classically equivalent to the Axiom of Choice. The proof is divided into two stages, of which the first is that which depends on applying Zorn's Lemma, and the second that which acts on the result of doing so. The collection of norm-preserving extensions

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & \mathbb{R} \\ \uparrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

of the linear functional $\varphi : A \to \mathbb{R}$ to linear functionals $\varphi' : A' \to \mathbb{R}$ defined on subspaces of B containing the subspace A, partially ordered by extension, is an inductive set, in which therefore, by Zorn's Lemma, there exist maximal members. Choosing such a maximal norm-preserving extension, which we shall denote by $\varphi' : A' \to \mathbb{R}$, we assert that the linear subspace A' on which the extension is defined is actually the seminormed space B, from which the theorem follows.

To see this, we argue by contradiction, assuming that there exists an element $b \in B$ which lies outside the subspace A'. In this case, it is possible to extend the linear functional

$\varphi' : A' \to \mathbb{R}$ to a linear functional $\varphi'' : A'' \to \mathbb{R}$ defined on the subspace obtained by extending A' by the element $b \in B$ by assigning to $b \in B$ an arbitrarily chosen element $\beta \in \mathbb{R}$. The criterion for the extension thus defined to be norm-preserving is that $\beta \in \mathbb{R}$ be chosen satisfying the condition that

$$\sup_{a \in A} \{ \varphi(a) - \|a - b\| \} \leq \beta \leq \inf_{a \in A} \{ -\varphi(a) + \|a + b\| \} .$$

It may be shown fairly easily that indeed one has that $\sup_{a \in A} \{ \varphi(a) - \|a - b\| \}$ is not greater than $\inf_{a \in A} \{ -\varphi(a) + \|a + b\| \}$, allowing the choice of such an element $\beta \in \mathbb{R}$, from which the result follows by contradiction of the maximality of the extension $\varphi' : A' \to \mathbb{R}$. Hence, there exists a norm-preserving extension to a linear functional $\psi : B \to \mathbb{R}$. Of course, we have been dealing here with the case of a seminormed linear spaces over the reals, from which the result for the complex case is easily deduced.

The objective that we have is to obtain a proof of the Hahn-Banach theorem which is constructively valid [22]. It is evident, from whatever constructive standpoint one may take, that the proof just outlined is not constructive. Most immediately, the application of the Axiom of Choice in the form of Zorn's Lemma renders the proof decidedly non-constructive. The method of proof by contradiction is also likely to be an obstacle to constructivity, although this is a difficulty which is quite often surmountable. More subtly, a dependence on the construction of suprema and infima in the set of real numbers will cause anxiety in the constructive mind, since $\mathbb{R}$ constructively is not order-complete. At another level of concern, there is the question of what exactly should be meant by the terms *seminormed linear space* and *bounded linear functional* within the constructive context, since these may classically be expressed in many syntactically different, but semantically equivalent, ways, which may no longer be semantically equivalent from a constructive viewpoint. Indeed, we may wish later to apply the same remark to the form of the *statement* of the Hahn-Banach theorem.

Before any of this, we should first specify precisely what we shall be meaning by *constructive*, and perhaps indicate why we are interested in finding a constructive form of the Hahn-Banach, or indeed any other, theorem. The answers to these questions are essentially the same: by *constructive mathematics* we shall mean mathematics that is valid in any topos having a natural number object. The motivation for deciding on this description is that it encompasses many situations that occur in practice, in terms of extending mathematics to cases involving questions of continuity in parameters, or of equivariance with respect to the action of a group or monoid. For a discussion of this particular viewpoint, the reader is referred to Mulvey and Pelletier [20] and to Banaschewski and Mulvey [4,6]. For the moment, suffice it to say that the category of sheaves on a topological space X and the category of M-sets for any monoid M are instances of the concept of a topos. For a leisurely discussion of the interpretation of mathematics in a category of sheaves, the reader is referred to Mulvey [17]. The formalities may be found in Johnstone [13].

Pragmatically, and to a great extent formally, this means that we may develop mathematics within a context in which we may still talk of sets and mappings, while limiting ourselves to constructions that are based on the formation of cartesian products of pairs of sets, of the power set of a set, and of the subset of a set consisting of elements satisfying a formula in the usual language of sets. Additionally, we may assume the existence of a set of natural numbers [13], satisfying the Peano axioms in the usual way. Logically, we may use all the usual rules of deduction of the Predicate Calculus, with the exception of those derived from the Law of the Excluded Middle, asserting of any proposition φ that the disjunction

$$\varphi \vee \neg\varphi$$

of the proposition with its negative is necessarily valid. In particular, we must make use neither of the Axiom of Choice, nor of any of its variants, such as Zorn's Lemma. Indeed, even the Axiom of Countable Choice, which allows the choosing of a sequence of elements from a sequence of sets, each having at least one element, is not valid within this context. It may be remarked in passing that, within a topos the validation of the Axiom of Choice implies that of the Law of the Excluded Middle [13].

It is already evident that the proof of the Hahn-Banach theorem outlined above fails to remain valid constructively on each and every of the grounds just considered. Indeed, within the slightly different context of the constructivity considered in intuitionistic mathematics, it became clear that there was little hope of pushing the conventional proof through in anything beyond the case of a seminormed space of countable dimension. In fact, even that case depends on the acceptance within intuitionistic mathematics of the Axiom of Countable Choice, and even the Axiom of Countable Dependent Choice, in which the choosing of each element of a sequence is dependent on the choices made thus far. The reader is referred to Bishop [8] and to Bridges [9] for various valiant attempts to push through some form of the Hahn-Banach theorem in greater generality, by requiring only that the extending functional is approximately equal to that which is being extended. For an insight into the desperation experienced in trying to achieve even this kind of progress, one should read Bishop's introductory comments on Heyting's endeavours in respect of the continuum [8].

One part of the difficulty that faced the intuitionistic mathematician seems to have arisen from a philosophical position which still tried to prove mathematical results in a form as close as possible to that considered classically, with little attempt being made to consider the meaning of the theorem, rather than simply its proof. There was also little attempt made to adapt the definitions involved in the mathematics being considered to the context in which that consideration was taking place. In the case of seminormed spaces, taking the naive definition of a seminorm leads fairly immediately to difficulties, for instance in a seminormed space admitting quotient spaces only by subspaces satisfying additional, non-intuitive, conditions [9]. Explicitly, the quotient space of a seminormed linear space B by a linear subspace A will not admit a seminorm unless the subspace is *located*, in the sense that the infimum

$$\inf_{a \in A} \|a + b\|$$

exists for each $b \in B$. Again, this is a difficulty that arises through the set of real numbers failing to be order-complete, in this case in the sense of intuitionistic constructivity.

To avoid difficulties of this kind, and to provide the conceptual setting within which one can begin to constructivise the Hahn-Banach theorem, means adapting the definition of the concept of seminormed linear space to the constructive context. In doing this, one may also be guided by insights into what seminormed spaces in certain toposes should turn out to be. In particular, one believes that seminormed, normed, and Banach spaces in the category of sheaves over a topological space X should correspond to bundles of seminormed, normed, and Banach spaces over X in the more conventional sense arising in parametrising these structures continuously over X. The detail of the elucidation of these concepts may be found elsewhere [11], together with a discussion of their equivalence with the corresponding concepts of bundles over the topological space X when interpreted in the category of sheaves on X. For the moment, it suffices to observe the following:

DEFINITION. By a *seminormed linear space* A will be meant a linear space over the field $\mathbb{Q}$ of rational numbers, together with a mapping

$$N : \mathbb{Q}^+ \to \Omega^A$$

from the set of positive rationals to the power set of the set A, satisfying the following conditions:

a) $0 \in N(q)$;

b) $a \in N(q) \to pa \in N(pq)$;

c) $a \in N(q) \wedge a' \in N(q') \to a + a' \in N(q + q')$;

d) $\exists\, q \in \mathbb{Q}^+ \; a \in N(q)$;

e) $a \in N(q) \leftrightarrow \exists\, q' < q \;\; a \in N(q')$,

for all $a, a' \in A$ and for all $p, q, q' \in \mathbb{Q}^+$.

It should be noted that in this and other definitions, those concepts not defined *ab initio* are to be interpreted just as in classical mathematics. Thus, the set of rational numbers is constructed from the set of natural numbers in the usual way, and the subset of positive rationals is carved out from it by interpreting the conventional formula expressing positivity [17].

The fact that we have considered linear spaces over the rational numbers, rather than the reals, will simplify the constructions at a later point, while extending slightly the scope of the Hahn-Banach theorem. More importantly, the axiomatisation of the notion of a seminorm is centred around the concept of the open balls which classically a seminorm determines. For each positive rational $q \in \mathbb{Q}$, the subset

$$N(q)$$

of the linear space A assigned by the seminorm mapping is to be thought of as the open ball of radius q. The axioms express the usual properties of a seminorm in terms of these open balls. Of course, classically one may retrieve a seminorm in the more conventional sense, by defining

$$\|a\| = \inf \{\, q \in \mathbb{Q}^+ \mid a \in N(q) \,\}$$

for each $a \in A$. The problem constructively is that such an infimum need not necessarily exist in the set $\mathbb{R}$ of real numbers. Indeed, by now one may be beginning to have some doubts about what exactly the set $\mathbb{R}$ of real numbers is, since this involves a far more sophisticated construction than that of the set $\mathbb{Q}$ of rational numbers. This question, and that of whether the seminorm may indeed be described in terms of a seminorm function, is one to which we must later return.

2. Extending the Hahn-Banach theorem.

Having defined the context, one may examine the limitations that this is known to impose upon the problem. In fact, these are severe, although they also contain the hint of a way forward. In a word, it may be shown that it is not possible to prove the Hahn-Banach theorem constructively in the form stated. Explicitly, one may determine precisely the conditions under which the Hahn-Banach theorem stated above, which we shall refer to as the *naive* Hahn-

Banach theorem, is valid in two contexts to which constructive methods may be applied. The first of these is that of the category of sheaves over a topological space X, the context in which the Hahn-Banach theorem for bundles of seminormed linear spaces over X may be examined. The result, due to Banaschewski, is the following:

Theorem 2.1. *In the category of sheaves on a topological space X, the naive Hahn-Banach theorem is valid if, and only if, the topological space X is extremally disconnected.*

It may be recalled that a topological space X is said to be *extremally disconnected* provided that the closure of every open subset is again open.

On the one hand, this extinguishes for once and for all any hope that the theorem might be constructively provable, since extremally disconnected topological spaces constitute a fairly extreme class of spaces. On the positive side, it may be observed that the toposes corresponding to these topological spaces are distinguished amongst other categories of sheaves on topological spaces by being those in which De Morgan's Law, the weakening of the Law of the Excluded Middle that asserts that

$$\neg\varphi \vee \neg\neg\varphi$$

is valid for any proposition φ, is validated. Tantalisingly, this is also the condition for the set $\mathbb{R}$ of real numbers in a topos to be order-complete. That this is the case is no surprise in the light of work by Burden [10] establishing an extension of the Hahn-Banach theorem which is valid in the category of sheaves over any topological space X.

The set $\mathbb{R}$ of real numbers that the statement of the Hahn-Banach theorem refers to is that obtained from the set $\mathbb{Q}$ of rational numbers by constructing the set of Dedekind cuts on the rationals, in the sense of the following:

Definition. By a *Dedekind cut* $x \in \mathbb{R}$ on the set $\mathbb{Q}$ of rational numbers is meant an ordered pair $(\check{x}, \hat{x})$ consisting of subsets of $\mathbb{Q}$ satisfying the following conditions:

a) $\exists\, p \in \check{x}$ and $\exists\, q \in \hat{x}$;

b) $p' < p \wedge p \in \check{x} \to p' \in \check{x}$ and $q \in \hat{x} \wedge q < q' \to q' \in \hat{x}$;

c) $p' \in \check{x} \to \exists\, p > p' \; p \in \check{x}$ and $q' \in \hat{x} \to \exists\, q < q' \; q \in \hat{x}$;

d) $p \in \check{x} \wedge q \in \hat{x} \to p < q$;

e) $p < q \to p \in \check{x} \vee q \in \hat{x}$.

It may be remarked that the set $\mathbb{R}$ of *Dedekind real numbers* thereby defined may be canonically given the algebraic and order structure normally associated with the set of real numbers. Moreover, there is a canonical inclusion

$$\mathbb{Q} \to \mathbb{R}$$

of the set of rational numbers in the set of real numbers, with respect to which, in the seminorm canonically determined by this order-preserving, algebraic embedding, the seminormed space of real numbers is exactly the completion of the seminormed space $\mathbb{Q}$. However, although classically this seminormed completion of the set $\mathbb{Q}$ of rational numbers is also order-complete, constructively this is not the case. Indeed, the equivalence of these conditions in any particular topos may be shown to be equivalent to the validation there of De Morgan's Law, in the sense referred to above.

There is, however, also a constructive order-completion $*\mathbb{R}$ of the set $\mathbb{Q}$ of rational numbers, obtained by a slight adaptation of a construction originally investigated by Staples [23] , and consisting of Dedekind *cuts on the rationals, in the sense of the following:

DEFINITION. By a *Dedekind *cut* $x \in *\mathbb{R}$ on the set $\mathbb{Q}$ of rational numbers is meant an ordered pair $(\underset{\vee}{x}, \hat{x})$ consisting of subsets of $\mathbb{Q}$ satisfying the conditions for a Dedekind cut, with the last condition replaced by the following:

$$\text{*f)} \qquad \exists\, p' > p\ (\forall\, q\ (q \in \hat{x} \ \rightarrow p' < q)) \rightarrow p \in \underset{\vee}{x} \quad \text{and}$$
$$\exists\, q' < q\ (\forall\, p\ (p \in \underset{\vee}{x} \ \rightarrow p < q')) \rightarrow q \in \hat{x}\ .$$

Once again, it may be remarked that the set $*\mathbb{R}$ of *Dedekind *real numbers* thereby defined may be canonically endowed with algebraic and order structure, together with a canonical inclusion

$$\mathbb{Q} \rightarrow *\mathbb{R}$$

of the set of rational numbers in the set of *real numbers, with respect to which, in the order canonically inherited from that of the set $\mathbb{Q}$, the partially-ordered set $*\mathbb{R}$ of *real numbers is exactly the completion of the partially-ordered set $\mathbb{Q}$. Moreover, it may straightforwardly be shown that the canonical inclusion of $\mathbb{Q}$ naturally factors through that of $\mathbb{Q}$ in the set $\mathbb{R}$ of Dedekind real numbers, yielding that $*\mathbb{R}$ is an extension of $\mathbb{R}$. Indeed, the inclusion

$$\mathbb{R} \rightarrow *\mathbb{R}$$

is an isomorphism exactly if De Morgan's Law is validated in the topos under consideration. In the absence of this non-constructive condition, the set $*\mathbb{R}$ is the natural completion of $\mathbb{R}$ to a partially ordered field in which both infima and suprema may be constructed.

Consideration of the set $*\mathbb{R}$ of Dedekind *real numbers provides an immediate solution to one of the problems already noted in connection with proving the Hahn-Banach theorem, namely the need, in establishing that a maximal extension chosen by Zorn's Lemma is necessarily defined on the seminormed space itself, to consider an extension obtained by assigning to an element lying outside the domain of the maximal extension a value satisfying

$$\sup_{a \in A} \{\, \varphi(a) - \|a - b\| \,\} \leq \beta \leq \inf_{a \in A} \{\, -\varphi(a) + \|a + b\| \,\}\ .$$

In the Dedekind *reals, these suprema and infima may be calculated, although it may then be necessary for the value chosen to be itself an element $\beta \in *\mathbb{R}$. Taking the lead from this observation that, in the constructive context, it may be necessary to consider *linear *functionals*

$$\psi : B \rightarrow *\mathbb{R}$$

on a seminormed linear space, instead of linear functionals, and applying the existence of an internal form of Zorn's Lemma in the category of sheaves on a topological space X, a delicate argument allowed Burden [10] to prove the following extension of the Hahn-Banach

theorem to this particular case:

THEOREM 2.2. *For any bounded linear *functional* φ *on a linear subspace* A *of a seminormed linear space* B *in the category of sheaves on a topological space* X, *there exists a norm-preserving extension to a linear *functional* ψ *defined on the seminormed space* B *itself.*

From any kind of constructive philosophical standpoint, this is still highly unsatisfactory, since it depends entirely on the Axiom of Choice in the context within which the sheaves are being defined. From a pragmatic standpoint, there are also serious drawbacks, notably that inherently there exist normed spaces that are non-zero, yet on which only the zero *functional is defined. Hence, any hope of applying this version of the Hahn-Banach theorem to establish a canonical embedding of any normed space in its double dual will be shortlived. For a discussion of these ideas the reader is nevertheless referred to Burden and Mulvey [11], since these linear *functionals occupy an interesting place in the investigation of seminormed bundles over topological spaces.

The import of this failed attempt at overcoming the difficulties introduced by such things as the absence of De Morgan's Law from the logic of constructive mathematics is to identify more definitely the problems caused by absence of an Axiom of Choice as those which must be tackled. That this is indeed the case becomes further evident on considering a second context within which limitations can be seen in trying to extend the naive Hahn-Banach theorem, namely that of the Hahn-Banach theorem in the category of M-sets for any monoid M. This is very different from the context just discussed in that the Law of the Excluded Middle, and hence, *a fortiori*, De Morgan's Law, is always validated in a topos of this kind, implying that the difficulties arise simply due to the invalidity of the Axiom of Choice in such a topos. The result, again due to Banaschewski [1] , is somewhat technical, although an examination of the proof shows clearly the reason why the condition concerned arises:

THEOREM 2.3. *In the category of* M*-sets for a monoid* M, *the naive Hahn-Banach theorem is valid if, and only if, the monoid* M *admits an invariant mean.*

It may be recalled that by an *invariant mean* μ on a monoid M is meant a linear functional

$$\mu :\ \mathbb{R}^b(M) \rightarrow \mathbb{R}$$

on the Banach space $\mathbb{R}^b(M)$ of bounded real functions on the monoid M, satisfying the conditions that

$$\mu(\mathbf{1}) = 1 \quad \text{and} \quad \mu(m\varphi) = \mu(\varphi) \quad \text{for all } m \in M \text{ and } \varphi \in \mathbb{R}^b(M)\ ,$$

in which $\mathbf{1} \in \mathbb{R}^b(M)$ denotes the constant function with value 1, and the action of $m \in M$ on $\varphi \in \mathbb{R}^b(M)$ is defined by $m\varphi(x) = \varphi(xm)$ for each $x \in M$.

In each of these cases, the problems introduced by applying the Axiom of Choice are clearly revealed. In the case of the Hahn-Banach theorem in the category of sheaves over a topological space X, although the classical Hahn-Banach theorem allows an extension of the linear functional induced on the fibre of the subspace at each point $x \in X$ to be chosen in a norm-preserving manner, the necessary arbitrariness of this choice, implied by the Axiom of

Choice being applied to do the choosing, means that one has no control over the coherence of these choices across the topological space X. The consequence is that although the linear functional originally given varies continuously across the space X, there is no chance of showing this to be the case for the extending functional, unless the topological space is extremally disconnected. It will be seen later that the extension of the Hahn-Banach theorem contained in the result of Burden may in fact be recovered by a purely geometric argument from that provable over an extremally disconnected space.

In the case of the Hahn-Banach theorem in the category of *M*-sets, the equivariant linear functional on the subspace admits, by the classical Hahn-Banach theorem, a multitude of norm-preserving extensions, chosen, by application of the Axiom of Choice, without concern for equivariance. The consequence is that there is no means of guaranteeing the existence amongst these of an equivariant extension, except in the case that there exists an invariant mean, in which case it may be applied to average out the action of the monoid on an arbitrarily chosen extension, allowing an equivariant extension to be constructed.

To tackle these situations properly, and to gain insight into the geometric aspects of avoiding applying the Axiom of Choice, requires a radically different approach to the problem.

3. Globalising the Hahn-Banach theorem.

One may make a start by considering objectively what exactly the Axiom of Choice contributes through allowing the proof of a theorem like the Hahn-Banach. Certainly, it allows the choice of a norm-preserving extension of a particular linear functional. In so doing, it establishes the existence of such an extension. All it tells you about the extension to be chosen is that it is a linear functional, that it is an extension of the given functional, and that it has the same norm. In fact, any particular choice contributes no more, and no less, than any other, or indeed the totality of all possible choices. Since such a choice may be made for any bounded linear functional on the given subspace, one is left in reality only with a statement that the mapping from the collection of all functionals defined on the seminormed space B to the collection of all functionals defined on the subspace A, given by restriction of domain of functionals, is surjective. Including the condition that the extension is norm-preserving is achieved by constraining this assertion to apply not to all linear functionals, but only to those functionals of norm not exceeding 1.

With this realisation comes another, that the set of linear functionals of norm not exceeding 1 on a seminormed linear space B is not just a set, but actually a topological space once endowed with the weak* topology. Indeed, by Alaoglu's theorem, the unit ball $\mathrm{Fn}\,B$ of the dual of a seminormed space B is a compact, Hausdorff topological space. Since the mapping given by restriction is indeed continuous, the Hahn-Banach theorem may be expressed exactly by saying that the restriction mapping

$$\mathrm{Fn}\,B \to \mathrm{Fn}\,A$$

from the unit ball of the dual space of the seminormed space B to the unit ball of the dual space of any subspace A is a quotient map of compact, Hausdorff spaces. Moreover, it may be remarked that this is frequently the form in which the Hahn-Banach theorem is applied, for instance to show that any normed linear space may be embedded isometrically in the Banach algebra of continuous real functions on a compact, Hausdorff topological space, namely the unit ball of its dual.

There is an immediate further connection with the Axiom of Choice to which these ideas point. One of its important applications is to establish the existence of the Stone-Čech compactification of any topological space. By work of Banaschewski and Mulvey [2,3], and independently by Johnstone [15], the rôle of the Axiom of Choice in proving this theorem is known precisely. The context for these constructive approaches to Stone-Čech compactification is that within which topological spaces generally are discussed in the absence of the Axiom of Choice, namely the theory of locales [16]. The idea is that constructively it is generally straightforward to create the lattice of open subsets of a topological space, even if the Axiom of Choice is needed to construct the set of points that are to be topologised. For instance, one approach to Stone-Čech compactification is to consider the maximal ideal space $\mathrm{Max}\,\mathbb{R}^b(X)$ of the algebra of bounded continuous real functions on the topological space X. The Axiom of Choice, once more in the form of Zorn's Lemma, is needed to establish the existence of maximal ideals of this algebra, although the lattice of open subsets of the maximal ideal space is known to be that of ideals of the algebra that are closed with respect to the canonical sup-norm [3]. In this and other situations, one is led to give prominence to the lattice of open subsets of the topological space, rather than to the set of points being topologised, a concept which is abstracted in the following:

Definition. By a *locale* L is meant a complete lattice in which the operations of finite meet and arbitrary join satisfy the condition that:

$$u \wedge \bigvee_i v_i = \bigvee_i u \wedge v_i$$

for any $u \in L$ and any elements $v_i \in L$.

Of course, any topological space will give rise to a locale, by considering its lattice of open subsets, but, constructively speaking, there are many situations in which one may define a locale, having the same properties as the locale of the topological space that would classically have been considered, but which is not necessarily the locale of a topological space, simply because in the absence of the Axiom of Choice the existence of its points cannot be established. Thus, it may be proved constructively that for any locale L there is a compact, completely regular locale βL together with a natural map of locales $L \to \beta L$ such that for any map of locales $L \to M$ to a compact, completely regular locale M there is a unique factorisation

$$\begin{array}{ccc} L & \longrightarrow & \beta L \\ & \searrow & \downarrow \\ & & M \end{array}$$

through the canonical map. It may be proved that in the presence of the Axiom of Choice every compact, completely regular locale is the lattice of open subsets of a compact, completely regular topological space [2]. Hence, classically this result is exactly the statement of the Stone-Čech compactification theorem, whilst constructively it is the appropriate formulation of that theorem. The concept of a map of locales met with here is that given by the following:

Definition. By a *map of locales*

$$\varphi : L \to M$$

is meant a mapping, referred to as the *inverse image mapping*,

$$\varphi^* : M \to L$$

that preserves finite meets and arbitrary joins.

The importance of locales constructively is that there is an extremely intuitive way of constructing them in practical situations. This arises from the observation that the lattices that underlie locales are, within constructive logic, exactly the Lindenbaum algebras of *propositional geometric theories* [20]. These are propositional theories generated starting from certain given primitive propositions by applying only the logical connectives of finite conjunction $\wedge$ and arbitrary disjunction $\bigvee$, of which the empty instances provide respectively the constants *true* and *false* . The axioms of such a theory are required to be entailments of the form

$$\varphi_1 \wedge \cdots \wedge \varphi_n \;\vdash\; \bigvee_i \; \psi_{i1} \wedge \cdots \wedge \psi_{in_i} \; ,$$

in which the symbols φ_i and ψ_{ij} denote primitive propositions of the theory. The Lindenbaum algebra of the theory is then the set of propositions

$$\bigvee_i \; \varphi_{i1} \wedge \cdots \wedge \varphi_{in_i}$$

of the theory, partially ordered by provable entailment in the theory, modulo provable equivalence in the theory. The fact that entailments are determined within constructive logic means precisely that the Lindenbaum algebra is a locale. Moreover, any locale is canonically isomorphic to the Lindenbaum algebra of an appropriate theory. It is perhaps significant that moving to constructive logic yields this abstraction of the concept of topological space, whilst remaining within classical logic gives only the much less interesting notion of a complete Boolean algebra.

The remark that any locale is canonically isomorphic to the Lindenbaum algebra of a propositional geometric theory is also of significance for our present purpose. In a quite precise sense, which may be found discussed at length elsewhere [20], the theory that is needed in order to obtain a particular locale is always the theory of its points. Presently, it will suffice to see this in the particular case of the locale that we wish to consider, namely that of the unit ball of the dual of a seminormed space B . To construct this locale, we need only find a propositional geometric theory of which the models within the constructive context are the linear functionals on B of norm not exceeding 1 . To find the primitive propositions, we may take a lead from the way in which classically we define the open subsets of the weak* topology that we place on this set of linear functionals. This topology is defined by taking a subbasis of open subsets given by subsets

$$U(a,r,s) \;=\; \{\, \varphi \in \mathrm{Fn}\, B \mid \varphi(a) \in (r,s) \,\}$$

consisting of all linear functionals of norm ≤ 1 which map a given $a \in B$ into the rational open interval (r,s) . Observe that taking these to be a subbasis of open subsets means that every open subset is an arbitrary join of finite meets of open subsets of this form.

Consider now the propositional geometric theory $\mathbb{F}\mathrm{n}\, B$ determined by taking a primitive proposition

$$a \in (r,s)$$

for each $a \in B$ and for each pair $r,s \in \mathbb{Q}$ of rational numbers, together with the following axioms:

(F1) $\quad true \;\vdash\; 0 \in (r,s) \quad$ whenever $\; r < 0 < s$;

(F2) $\quad 0 \in (r,s) \;\vdash\; false \quad$ otherwise ;

(F3) $\quad a \in (r,s) \;\vdash\; -a \in (-s,-r)$;

(F4) $\quad a \in (r,s) \;\vdash\; ta \in (tr,ts) \quad$ whenever $\; t > 0$;

(F5) $\quad a \in (r,s) \wedge a' \in (r',s') \;\vdash\; a + a' \in (r+r', s+s')$;

(F6) $\quad a \in (r,s) \;\vdash\; a \in (r,s') \vee a \in (r',s) \quad$ whenever $\; r < r' < s' < s$;

(F7) $\quad true \;\vdash\; a \in (-1,1) \quad$ whenever $\; a \in N(1)$;

(F8) $\quad a \in (r,s) \;\dashv\vdash\; \bigvee_{r<r'<s'<s} a \in (r',s')$.

Denoting the locale obtained from this theory by $\mathrm{Fn}\, B$ it may be verified [20] that its models are indeed exactly the linear functionals on B of norm ≤ 1 . The correspondence between models Φ and functionals φ is given by

$$\Phi \vDash a \in (r,s) \quad \text{if and only if} \quad \varphi(a) \in (r,s) \; .$$

In particular, the linear functional determined by a particular model of the theory assigns to each $a \in B$ the Dedekind real number of which the lower and upper cuts consist respectively of those rational numbers $r,s \in \mathbb{Q}$ for which the primitive proposition $a \in (r,s)$ is validated in the model. The axioms of the theory are exactly those required to prove that these are indeed the lower and upper cuts of a Dedekind real number, and that the assignment to each $a \in B$ of this Dedekind real number yields a linear functional on the seminormed space B of norm not exceeding 1 . In this way, the axiomatisation of the theory is almost self-evident once one recalls the relationship classically between the points of the space to be constructed and the description of the topology that classically is placed upon them.

Recalling the motivating remarks concerning the expression of the Hahn-Banach theorem in terms of the continuous mapping from the unit ball of the dual of the seminormed space B to that of the dual of the subspace A , the Hahn-Banach theorem may be stated constructively in the following way:

Theorem 3.1. *For any linear subspace A of a seminormed linear space B , the map*

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

canonically induced by restriction of functionals is a quotient map of compact, completely regular locales.

For the moment, we shall indicate a geometric proof of that theorem in a context that is not entirely constructive. Explicitly, we shall show that the theorem may be proved in any Grothendieck topos, that is, a topos that is bounded over the topos of sets [13], by a geometric argument that globalises the theorem from its classical counterpart in the topos of sets. The details of the proof may be found in the paper by Mulvey and Pelletier [20], in which the construction used is explained with some care. The view taken is that the technique applied is one which exactly shows how questions of continuity in parameters and of equivariance should be addressed constructively. In a sense, the implication is that this captures the constructive aspects of globalisation, even in the case where it is unknown whether the theorem to be globalised is indeed constructively provable. In the present case, we shall see later that the Hahn-Banach theorem may in fact be proved constructively in the form now introduced, thereby establishing a stronger assertion than that which we now consider. In each case, however, the map of locales about which the assertion is made is that given by interpreting the theory obtained from the linear subspace A in that obtained from the seminormed linear space B, by assigning to each primitive proposition

$$a \in (r, s)$$

of the theory $\mathbb{F}\mathrm{n}\, A$ the corresponding proposition of the theory $\mathbb{F}\mathrm{n}\, B$. The assignment determines a map of locales

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

provided that each of the axioms of the theory $\mathbb{F}\mathrm{n}\, A$ is validated in the locale $\mathrm{Fn}\, B$, which is trivially the case since each such axiom is indeed one of the axioms of the theory $\mathbb{F}\mathrm{n}\, B$. The condition for the map of locales thereby defined to be a quotient map is exactly that its inverse image mapping is an embedding. In terms of the theories, this exactly states that the theory $\mathbb{F}\mathrm{n}\, B$ is a *conservative extension* of the theory $\mathbb{F}\mathrm{n}\, A$, an assertion which can be summarised by saying that one can learn no more in terms of linear functionals about the subspace A by considering it embedded in the seminormed space B than one learn by considering it as a seminormed space in its own right, an observation which is frequently used to summarise the content of the classical Hahn-Banach theorem.

The proof that we shall outline depends critically on the fact that the locales concerned are indeed compact, completely regular locales. That this is the case may indeed be proved constructively, and the details of one approach may be found in Mulvey and Pelletier [20]. The complete regularity is deduced from that of the locale of real numbers, itself defined by introducing a propositional geometric theory of the Dedekind real numbers with completely regularity deduced from the geometry of the set of rational numbers. The compactness is established by a syntactic argument, based on the compactness of the locale of any propositional geometric theory containing only finitary disjunctions. Although in this context this approach is extremely unintuitive, the technique is borrowed from another context in which the intuitive content of the method is both apparent and meaningful [3]. The statement that the locale

$$\mathrm{Fn}\, B$$

is compact, completely regular for any seminormed linear space B is exactly Alaoglu's theorem, which is thereby established to be constructively valid in this localic form [20].

The Hahn-Banach theorem is now proved in any Grothendieck topos $\mathbb{E}$ by constructing another Grothendieck topos $\mathbb{B}$, in which the Axiom of Choice is validated, together with a geometric map

$$\gamma : \mathbb{B} \to \mathbb{E}$$

which is a covering of the topos $\mathbb{E}$. The topos $\mathbb{B}$, known as the Barr covering [7] of the topos $\mathbb{E}$, is the category of sheaves on a complete Boolean algebra constructed from the topos $\mathbb{E}$. In such a topos, the mathematics is classical due to the presence of the Axiom of Choice. In particular, the dual locale of any seminormed space in the topos is actually the lattice of open subsets of a compact, completely regular topological space, hence necessarily that of the unit ball of the dual of the seminormed space in the classical sense. As a consequence, the classical Hahn-Banach theorem indeed implies that its constructive form is also valid for seminormed spaces in the topos $\mathbb{B}$. This may now be used to infer that the constructive form of the Hahn-Banach theorem is also valid in the topos $\mathbb{E}$ in which we are seeking to prove the theorem. That this is the case is argued in the following way.

Consider the inverse image along the geometric covering

$$\gamma : \mathbb{B} \to \mathbb{E}$$

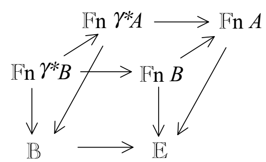
of the inclusion map from the linear subspace A to the seminormed linear space B. This yields an inclusion map

$$\gamma^* A \to \gamma^* B$$

of seminormed spaces in the topos $\mathbb{B}$. It should be noted in passing that it is to facilitate this step that we have considered linear spaces over the field $\mathbb{Q}$ of rational numbers, since the field of rational numbers is preserved under inverse image, whereas that of Dedekind real numbers is not. By the functoriality, straightforwardly verified, of the construction of the dual locale of a seminormed space, this yields a map

$$\mathrm{Fn}\, \gamma^* B \to \mathrm{Fn}\, \gamma^* A$$

of locales in the topos $\mathbb{B}$. Denoting the toposes of sheaves on each of the locales constructed by the corresponding open faced symbols, the functoriality of the construction of toposes of sheaves yields the following commutative diagram of toposes and geometric maps:



Since the axiomatisation of the theory of the dual locale is geometric, it is preserved under inverse image along a geometric map, hence the topos $\mathbb{F}\mathrm{n}\, \gamma^* B$ is actually the pullback of the topos $\mathbb{F}\mathrm{n}\, B$ along the geometric map

$$\gamma : \mathbb{B} \to \mathbb{E}.$$

Moreover, by the compact, complete regularity of the locale $\mathrm{Fn}\, B$, the fact that this geometric map is actually a covering implies that its inverse image

$$\mathbb{F}\mathrm{n}\, \gamma^* B \to \mathbb{F}\mathrm{n}\, B$$

along the canonical map from $\mathbb{F}\mathrm{n}\, B$ to the topos $\mathbb{E}$ is again a covering. Furthermore, similar remarks apply to the geometric maps between the toposes determined by the linear subspace A. Finally, by the remark that the Hahn-Banach theorem applies classically in the topos $\mathbb{B}$, the map

$$\mathrm{Fn}\, \gamma^* B \to \mathrm{Fn}\, \gamma^* A$$

considered above is actually a quotient map of locales, hence determines a covering map of the corresponding toposes of sheaves. Hence, considering the commutative square that sits on top of the diagram of toposes, we observe that each of the maps concerned, with the exception of the map

$$\mathbb{F}\mathrm{n}\, B \to \mathbb{F}\mathrm{n}\, A,$$

has already been shown to be a geometric covering, from which it follows that this map of toposes is also a covering. However, the geometric map determined by a map of locales is a covering exactly if the map of locales is a quotient map. Hence, the canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

is a quotient map of locales, establishing [20] the Hahn-Banach theorem in the topos $\mathbb{E}$.

Although this account has depended on a considerable amount of background from the theory of toposes, it has served to make the point that, once the Hahn-Banach theorem has been presented in a form appropriate to the constructive context, globalising the theorem to situations which include those of categories of sheaves on topological spaces, and of categories of sets acted upon by a monoid, is a matter of straightforward geometric argument involving covering toposes. The next stage of the development of these ideas involves in some sense simplifying these ideas to the point at which the argument may be carried out within the topos in which the Hahn-Banach theorem is needed. In other words, making the geometric argument self-contained, and thereby entirely constructive.

Before doing so, however, it may be of interest to apply the theorem just proved, and the technique used to prove it, to retrieving the extension of the naive Hahn-Banach theorem considered in the preceding section. So then, suppose that we are given a bounded linear *functional

$$\varphi : A \to {}^*\mathbb{R}_{\mathbb{E}}$$

on a linear subspace A of a seminormed space B in the topos $\mathbb{E}$ of sheaves on a topological space X. Recall that there exists another covering

$$\alpha : \mathbb{G} \to \mathbb{E}$$

of any topos $\mathbb{E}$, referred to as the Gleason covering [14], by a topos $\mathbb{G}$ in which De Morgan's Law is validated. In so doing, we should also remark that, unlike that of the covering considered above, the construction of this covering is entirely constructive. It has already been remarked that in any topos satisfying De Morgan's Law, the Dedekind real numbers are already order-complete, hence coincide with the Dedekind *real numbers. One of the properties of the Gleason covering [14] is that the direct image $\alpha_* \mathbb{R}_{\mathbb{G}}$ of the set of Dedekind real numbers in the topos $\mathbb{G}$ is exactly the set of Dedekind *real numbers ${}^*\mathbb{R}_{\mathbb{E}}$ in the topos $\mathbb{E}$. Hence, the bounded linear *functional considered above may in fact be written in the form

$$\varphi : A \to \alpha_* \mathbb{R}_{\mathbb{G}}.$$

By the adjointness of inverse image and direct image functors, this therefore determines a bounded linear functional

$${}^*\varphi : \alpha^* A \to \mathbb{R}_{\mathbb{G}}$$

on the inverse image $\alpha^* A$ of the seminormed space A, hence a point of the locale $\mathrm{Fn}\, \alpha^* A$ in the topos $\mathbb{G}$. By the constructive form of the Hahn-Banach theorem, just proved to be valid in the topos $\mathbb{G}$, the canonical map

$$\mathrm{Fn}\, \alpha^* B \to \mathrm{Fn}\, \alpha^* A$$

is a quotient map of locales. By a result of Mulvey and Pelletier [19], for any quotient map of compact, completely regular locales in a topos satisfying De Morgan's Law, any point of the locale $\mathrm{Fn}\, \alpha^* A$ may then be lifted to a point

$${}^*\psi : \alpha^* B \to \mathbb{R}_{\mathbb{G}}$$

of the locale $\mathrm{Fn}\, \alpha^* B$, which by adjointness yields the required extension

$$\psi : B \to {}^*\mathbb{R}_{\mathbb{E}}$$

of the given bounded linear *functional.

4. Constructivising the Hahn-Banach theorem.

The approach to the constructive Hahn-Banach theorem described above relies on an application of geometric constructions at a contextual level to globalise the Hahn-Banach theorem to situations in which the Axiom of Choice cannot be applied. This technique may be extremely valuable in certain situations, of which the recovery of the extended version of the naive Hahn-Banach theorem is an instance. Although it gives some metamathematical credence to the view that the constructive Hahn-Banach theorem may indeed be provable constructively, it does not in itself lead to a constructive proof. To achieve this, one needs to be more careful in analysing the classical proof of the theorem, and the way in which its apparent dependence on the Axiom of Choice may be replaced by more subtle geometric arguments. Certainly, one aspect of the proof just outlined that is likely to need to be incorporated in arriving at such a constructive proof is that of the rôle played by the compact, complete regularity of the dual locale of a seminormed linear space.

This analysis, of which we shall give only an outline, involves working geometrically with locales, both in order to construct the locales that appear in the Hahn-Banach theorem itself, but also those needed to establish a constructive proof of the theorem. The principle that may be drawn from this is that careful geometrical argument can be used to avoid applications of the Axiom of Choice, except any that we may unwisely build into our mathematics by choosing to state a theorem in a hopelessly unconstructive way. For the Hahn-Banach theorem, this means that expressing the theorem in the form of a statement that the canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

of locales is a quotient is a constructively appropriate statement that admits a constructive proof. In the event that the locales concerned are indeed topological spaces, then the Hahn-Banach theorem in its classical form will also be valid. However, the statement that any compact, completely regular locale is indeed spatial is a statement almost equivalent to the Axiom of Choice. The details of the proof to be outlined will appear in a paper by Mulvey and Vermeulen [22], depending heavily on earlier work by Vermeulen [25].

The concept behind the proof is straightforward. The classical proof of the Hahn-Banach theorem provides an inductive step, albeit for the moment not in a constructive form, by which it is shown that a bounded linear functional defined on a subspace of a seminormed space may be extended in a norm-preserving manner to the subspace obtained by adjoining a single element of the seminormed space. The classical proof applies this to show that a maximal norm-preserving extension is necessarily defined on the seminormed space itself. In the constructive context, instead of applying Zorn's Lemma to choose such a maximal extension, we must instead use this proof of the existence of an extension to a subspace obtained by adjoining a single element, adapted to a constructive form, as the inductive step of an inductive limit argument carried out over the finitary extensions of the given functional. The inductive step involves showing that the corresponding map of dual locales is indeed not just a quotient, but a kind of map known as a *proper surjection* [25]. That this is the case stems from the compact, complete regularity of the dual locales concerned, allowing a geometric result of Vermeulen concerning filtered limits of proper surjections [25] to be applied to prove that the canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

is again a proper surjection, and hence a quotient map of locales. The subtler points of the proof are those involved in making the inductive step constructive, to which we shall turn after outlining the inductive framework within which that step will be applied.

Consider, then, a linear subspace A of a seminormed linear space B. Observe firstly that it is classically, and constructively, the case that the extension

$$A \to B$$

may be obtained as a filtered inductive limit of finitely generated extensions

$$A \to A[b_1, \cdots, b_n]$$

of the subspace A to the subspace $A[b_1, \cdots, b_n]$ of B obtained by adjoining finitely many elements $b_1, \cdots, b_n \in B$. This is evidently the case algebraically, and carries through without problem to the seminormed case. Intuitively, the idea of the proof would now be to establish, by applying the inductive step of the classical Hahn-Banach theorem, that, for the locales obtained by applying the construction of the dual locale to these seminormed spaces, the canonical map

$$\mathrm{Fn}\ A[b_1, \cdots, b_n] \to \mathrm{Fn}\, A$$

of locales is indeed a proper surjection. The canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

would then be the filtered limit of these proper surjections, hence again a proper surjection, and in particular a quotient map, yielding the constructive Hahn-Banach theorem.

Constructively, in order to apply the inductive step of the classical Hahn-Banach theorem straightforwardly, we have to be slightly more subtle. The problem is one that is often met with constructively, namely that for a given element $b \in B$ by which we might wish to extend a subspace A, it may not be decidable whether $b \in A$ or $\neg(b \in A)$. To avoid this

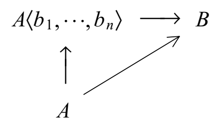
difficulty, which can be surmounted in a number of ways, we instead modify the inductive system considered by presenting the extension

$$A \to B$$

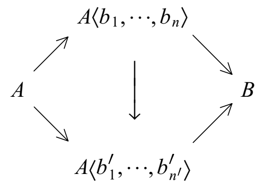
as the inductive limit of a filtered diagram of finitely freely generated extensions of the subspace A. Explicitly, consider for each finite family $b_1, \cdots, b_n \in B$ the extension

$$A \to A\langle b_1, \cdots, b_n\rangle$$

of the seminormed linear space A obtained by freely adjoining indeterminates representing these elements to the seminormed space A, together with the map of extensions



obtained by mapping each of these indeterminates to the corresponding element of the seminormed space B. Now consider the filtered diagram of extensions of the seminormed space A obtained by taking as maps those maps of extensions for which the diagram



commutes. Extending the intuitive argument, it is evident that the extension

$$A \to B$$

is the filtered inductive limit of this diagram in the category of extensions of the seminormed space A. Applying the syntactic construction of the dual locale of a seminormed space, the functoriality of the construction yields a corresponding filtered diagram in the category of locales over the locale $\mathrm{Fn}\, A$, of which it may formally be verified that the limit is the canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A\, .$$

Applying the theorem mentioned above concerning filtered limits of proper surjections of locales [25], it follows that this is again a proper surjection, hence in particular a quotient map, provided that it may be shown that the canonical map

$$\mathrm{Fn}\ A\langle b_1, \cdots, b_n\rangle \to \mathrm{Fn}\, A$$

is a proper surjection for each finite family $b_1, \cdots, b_n \in B$. Since the composite of proper surjections is a proper surjection, it suffices to prove this in the case of an extension freely generated by a single element $b \in B$, which brings us to the inductive step in the theorem.

In the classical proof of the Hahn-Banach theorem, the inductive step involves showing that for any bounded linear functional

$$\varphi : A \to \mathbb{R}$$

on a subspace A of a seminormed space B and for any element $b \in B$ the inequality

$$\sup_{a\in A} \{ \varphi(a) - \|a - b\| \} \leq \inf_{a\in A} \{ -\varphi(a) + \|a + b\| \}$$

is satisfied, allowing a norm-preserving extension of the functional to be defined on the subspace obtained by adjoining the element $b \in B$ to the subspace A. In the constructive case, the same ideas are used to show that the canonical map

$$\mathrm{Fn}\, B \to \mathrm{Fn}\, A$$

is a proper surjection whenever the seminormed space B is that obtained by freely adjoining an indeterminate $b \in B$ to the seminormed space A. To prove this, we observe firstly that one may construct a locale $\mathbb{R}$ of real numbers by taking a propositional geometric theory of which the primitive propositions are of the form:

$$x \in (r, s)$$

for each pair of rational numbers $r, s \in \mathbb{Q}$, together with axioms that describe that these rationals lie respectively in the lower and upper cuts of a Dedekind real number.

One may also construct locales $\mathbb{R}_L$ and $\mathbb{R}_U$ of lower and upper real numbers by taking propositional geometric theories describing respectively lower and upper cuts on the rationals, together with a sublocale

$$\sqsubseteq \to \mathbb{R}_L \times \mathbb{R}_U$$

of the product $\mathbb{R}_L \times \mathbb{R}_U$ obtained by adjoining the axiom

$$p < \sigma \wedge \tau < q \vdash p < q$$

to the axioms for the generic lower and upper cuts σ and τ. Equally, within the product $\mathbb{R}_L \times \mathbb{R} \times \mathbb{R}_U$ one may construct a sublocale

$$I \to \mathbb{R}_L \times \mathbb{R} \times \mathbb{R}_U$$

which classifies triples (σ, α, τ) having the property that (σ, τ) lies in the sublocale defined above, and that $\sigma \leq L(\alpha)$ and $U(\alpha) \leq \tau$, in which $L(\alpha)$ and $U(\alpha)$ denote respectively the lower and the upper cuts of a Dedekind real number α. It must be remembered throughout that these discussions are considered at a syntactic level, rather than in terms of actual Dedekind cuts on the rationals.

There is a canonical map

$$I \to \sqsubseteq$$

obtained by factoring the projection, and this map of locales may be shown to be a proper surjection. It is this fact, derived from the compactness properties of closed bounded intervals of the locale of reals, that lies at the heart of the Hahn-Banach theorem. Given this observation,

of which the proof may be found elsewhere [22], the required result may be proved by remarking that there is a canonical diagram

$$\begin{array}{ccc} \mathrm{Fn}\, B & \to & I \\ \downarrow & & \downarrow \\ \mathrm{Fn}\, A & \to & \sqsubseteq \end{array}$$

defined by encoding syntactically the inequality between the suprema and infima involved in the construction of an extension, described above in the case of the classical Hahn-Banach theorem. Moreover, this diagram is actually a pullback in the category of locales. In consequence, the fact that the right hand side is a proper surjection implies that the canonical map determined by the extension of seminormed spaces is also a proper surjection, which gives the required inductive step, and with it a constructive proof of the Hahn-Banach theorem stated in the preceding section.

In conclusion, it should be noted that the constructive proof has depended on simple manipulation of geometric objects, taken in conjunction with the deeper geometric observations of Vermeulen concerning the foundations of descent theory in the context of closed maps of locales [25]. In a sense, together these provide the geometric situation in which dependence on the Axiom of Choice is relegated to the question of whether the dual locales considered are in fact topological spaces, in which case the Hahn-Banach theorem can be recovered in its classical form. However, as may be found in papers by Mulvey and Pelletier [20,21], there is no need to have the Hahn-Banach theorem in the classical form, when it may be applied directly in the constructive form to yield the same consequences. Nevertheless, in the presence of the Axiom of Choice, the constructive form immediately yields the classical one, just by the observation that the locales considered are then exactly the lattices of open subsets of their topological spaces of points.

5. Conclusions.

In this paper, we have considered one particular application of a technique that seems to have quite widespread relevance. The Hahn-Banach theorem classically has been seen to be a spatialisation of a constructive theorem of which the proof is closely linked to the geometric content of the classical theorem. The geometry of the Axiom of Choice has been seen to be that which is founded on the notion of point as a primitive construct. Instead, the constructive geometry of locales, in which points have little relevance, with locality taking the primitive place, has appeared naturally to signify the extent to which we consider points as an ideal realisation of an otherwise syntactic expression of mathematical relationships. The geometry of constructive logic has been seen as a fundamental component of the constructivisation of mathematical concepts. In particular, through the link between locales and constructive logic, the geometry of locales provides a mathematical expression of metamathematical constructs.

The ideas behind these techniques have already been applied in a number of other situations. The construction of the Stone-Čech compactification of a locale is one instance that predates the work described here, with a variety of approaches leading to the existence of compactifications of locales both to compact, regular and to compact, completely regular locales. One of the approaches taken, establishing the existence of the Stone-Čech compactification by developing a constructive version of the Tychonoff theorem [15], has led to further work extending the constructive form of the Tychonoff theorem to compact locales [24]. The constructive form of Gelfand duality has been investigated, thereby stimulating the development of an extension towards the non-commutative case through the introduction of the theory of quantales [18]. The Fundamental Theorem of Galois theory has also been shown to have a constructive expression [12].

In each of these cases, the rôle played by the Axiom of Choice has been marginalised to one of recovering a familiar, but not necessarily more advantageous, form of the theorem from a version that gives constructive insight into the mathematics that lies behind it. In achieving this, constructive mathematics itself has moved forward from being involved with managing to achieve a proof against the odds, to describing positively the fundamentals of the mathematics that is being investigated. The importance of this step is also measured in terms of the applications of these ideas to situations within mathematics in which the formalism of constructive mathematics is needed to extend a classical theorem in another context, whether involving continuity across a space or equivariance with respect to an action of some kind.

The importance of the Axiom of Choice in allowing the rapid development of mathematics would appear to be justified by exactly the arguments being advanced here, pointing to the extent to which meaningful results within classical mathematics may be shown to carry through to the constructive context. But the message is also that with techniques that allow this more meaningful examination of mathematical structures increasingly at hand, whether one moves towards greater constructive insight or languishes in the perhaps less meaningful power of an approach to mathematics that may actually obscure insight into what is really happening is now at least largely a matter of choice.

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