A Quantisation of the Calculus of Relations

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The calculus of relations provides a basic tool for describing and manipulating specifications of programs of a machine. A specification of a program may be represented by a relation R on the set X of states of the machine. The programs which satisfy this specification are those having the property that, if $y \in X$ is a terminal state corresponding to running the program with initial state $x \in X$, then $(x, y) \in R$. The fact that the set $Q(X)$ of relations on a set X, partially ordered by inclusion of relations as subsets of the product $X \times X$, admits arbitrary joins, given by unions of subsets of the product, allows the construction of the weakest specification satisfied by any class of programs. This weakest specification may be obtained simply by taking the join of all specifications satisfied by each program in the class. In particular, it follows that, for any program Q and any specification R, there exists a weakest specification with the property that it is satisfied by precisely those programs P for which the program $P;Q$, namely P followed by Q, satisfies the specification R. The specification obtained in this way, denoted $Q \backslash R$, is called the weakest prespecification of the specification R given the program Q. This construction, together with others concerning the calculus of relations in the context of specifications of programs, may be found in the fundamental paper of Hoare and He [3].

The present paper shows that these constructions can be carried through to the case in which the behaviour of the machine is quantised, by replacing the set X of states by a Hilbert space H of states, by applying the concept of a quantale, introduced in [4]. In making this generalisation, our motivation initially is no more than the realisation that in other non-deterministic situations the introduction of this quantisation, in which the lattice of subsets of the set X is replaced by that of closed subspaces of the Hilbert space H, is extremely productive. However, the structures that begin to appear once this generalisation is allowed to evolve indicate that there may already exist intrinsic reasons for extending the calculus of relations in this way. In particular, there emerges a natural relationship with the concept of Gelfand quantale [6] arising in the spectral theory of C^*-algebras [7] which appears to merit further investigation. This, together with the appearance of certain operations encountered already within the context of linear logic, then becomes the primary motivation for considering the quantisation of the calculus of relations which will be introduced.

1. QUANTALES OF RELATIONS

DEFINITIONS. By a *quantale* Q will be meant a lattice having arbitrary joins $\bigvee$ together with an associative product & satisfying

$$a \,\&\, \bigvee b_i = \bigvee a \& b_i$$

and

$$\bigvee a_i \,\&\, b = \bigvee a_i \& b$$

for all $a, b, a_i, b_i \in Q$. The quantale Q will be said to be *unital* provided that there exists an element $e \in Q$ for which

$$e \& a = a = a \& e$$

for all $a \in Q$.

By an *involutive quantale* will be meant a quantale Q together with an involution; that is, with a unary operation *, satisfying

$$a^{**} = a$$
$$(a \& b)^* = b^* \& a^*$$

and

$$(\bigvee a_i)^* = \bigvee a_i^*$$

for all $a, b, a_i \in Q$. In the event that Q is also unital, then necessarily

$$e^* = e.$$

By a *homomorphism of quantales* will be meant a $\bigvee$- and &-preserving mapping

$$\varphi : Q \to Q'.$$

The homomorphism φ will be said to be *unital* provided Q and Q' are unital and

$$\varphi(e) = e',$$

for e and e' the respective units of Q and Q'. If Q and Q' are involutive quantales, the homomorphism φ will be said to be a *homomorphism of involutive quantales* if φ preserves the involution.

EXAMPLES. 1. Any locale L provides a trivial example of a unital involutive quantale, by taking the lattice L together with the product given by the meet $\wedge$, with unit given by the top element 1, and by defining $a^* = a$ for all $a \in L$.

2. Any C^*-algebra A with unit determines a unital quantale $\operatorname{Max} A$ by taking $\operatorname{Max} A$ to be the lattice of closed linear subspaces of A together with the product given by

multiplication of subspaces followed by closure. The quantale $\operatorname{Max} A$ is also involutive with involution defined by

$$M^* = \{x^* \in A \mid x \in M\}$$

for any closed linear subspace M of A.

3. The set $\mathcal{Q}(X)$ of relations on a set X, partially ordered by inclusion of relations, is a unital quantale with multiplication given by

$$R\&S = \{(x,y) \in X \times X \mid \exists\, z \in X \ (x,z) \in R \text{ and } (z,y) \in S\}$$

for any $R, S \in \mathcal{Q}(X)$, and with the equality relation providing the unit. Moreover, $\mathcal{Q}(X)$ is involutive, with involution defined by

$$R^* = \{(y,x) \in X \times X \mid (x,y) \in R\}.$$

The quantale L obtained trivially from a locale, the quantale $\operatorname{Max} A$ which is the spectrum of a C*-algebra A, and the quantale $\mathcal{Q}(X)$ of relations on a set X are each instances of the concept of a Gelfand quantale, in the sense now described:

Definitions. An element $a \in Q$ of a quantale Q is said to be *right-sided* if

$$a\&b \leq a$$

for all $b \in Q$. Similarly, $a \in Q$ is said to be *left-sided* if $b\&a \leq a$ for all $b \in Q$. The quantale Q is said to be *right-sided* (respectively, *left-sided*) if every $a \in Q$ is right-sided (respectively, left-sided). We observe that the subsets of right-sided and of left-sided elements of Q, $R(Q)$ and $L(Q)$ respectively, are quantales with respect to the operations of the quantale Q.

A quantale Q will be said to be a *Gelfand quantale* provided that Q is unital, involutive, and satisfies the condition

$$a\&a^*\&a = a$$

for all $a \in R(Q)$. We observe that this is equivalent to asking that

$$a\&a^*\&a = a$$

for all $a \in L(Q)$, since $a \in R(Q)$ if, and only if, $a^* \in L(Q)$.

For a C^*-algebra A the right-sided elements of the quantale $\operatorname{Max} A$ are the closed right ideals of A and the left-sided elements are the closed left ideals of A. The quantale $\operatorname{Max} A$ is, moreover, the motivating example of a Gelfand quantale, first introduced by Mulvey [5] to provide a concept of the spectrum for non-commutative C^*-algebras (see also [6,7]).

To see that $\operatorname{Max} A$ is a Gelfand quantale, let I be a closed right ideal and let $x \in I$. By the construction of approximate units in A, there exists a sequence (e_n) of self-adjoint elements of I for which

$$e_n x \to x$$

in A. The closed linear subspace U generated by the (e_n) obtained for each $x \in I$ then satisfies

$$U^* = U \leq I \leq U\&I$$

by definition. Hence,

$$I \leq U\&I \leq U\&U\&I \leq I\&I^*\&I \leq I,$$

since I is a right ideal, so $I = I\&I^*\&I$.

In the case of $\mathcal{Q}(X)$, the quantale of relations on the set X, we observe that the right-sided elements are those relations R_A for $A \subseteq X$ given by

$$R_A = \{(x,y) \in X \times X \mid x \in X - A\}.$$

Clearly, relations of this form are right-sided. Moreover, if S is any right-sided relation, we must have

$$S\&1 \leq S,$$

where 1 is the relation $\{(x,y) \in X \times X \mid x \in X,\ y \in X\}$. Hence, $S = R_A$, where

$$X - A = \{x \in X \mid (x,y) \in S \text{ for some } y \in X\}.$$

Similarly, the left-sided elements of $\mathcal{Q}(X)$ are those relations L_A of the type

$$L_A = \{(x,y) \in X \times X \mid y \in A\}.$$

It is the observation that $R_A^* = L_{X-A}$ and $L_A^* = R_{X-A}$ that permits us to establish the theorem:

Theorem 1. *For any set X, the quantale $\mathcal{Q}(X)$ of relations on X is a Gelfand quantale.*

Given an involutive quantale Q, the quantale $R(Q)$ of right-sided elements of Q can be endowed with a negation operation $\perp$ by defining

$$a^\perp = \bigvee_{a^*\&b=0} b$$

for $a, b \in R(Q)$; that is, $a^\perp$ is the largest element of $R(Q)$ such that

$$a^*\&a^\perp = 0.$$

It is easy to see that

$$a \leq a^{\perp\perp}$$

and that

$$(\bigvee a_i)^{\perp} = \bigwedge a_i^{\perp}$$

for all $a, a_i \in R(Q)$. Similarly, $L(Q)$, the quantale of left-sided elements of Q, also possesses a negation operation $\perp$ defined by

$$^{\perp}b = \bigvee_{a\&b^*=0} a$$

for $a, b \in L(Q)$; $^{\perp}b$ is the largest element of $L(Q)$ such that

$$^{\perp}b\&b^* = 0.$$

For any right-sided element R_A of $Q(X)$ one observes that

$$R_A^*\&R_{X-A} = \varnothing,$$

and that R_{X-A} is the largest relation S having the property that

$$R_A^*\&S = \varnothing.$$

Hence,

$$R_A^{\perp} = R_{X-A}.$$

Similarly, one observes that

$$^{\perp}L_A = L_{X-A}.$$

In general the negations on $R(Q)$ and $L(Q)$ do not provide a true complement in the lattice sense. However, there is a connection in the case of a Gelfand quantale between the notion of a negation and that of a complement, as is demonstrated by the Corollary to the following Proposition:

PROPOSITION. *For any Gelfand quantale Q, the quantale $R(Q)$ of right-sided elements has the following properties:*

(i) $a\&a = a$

(ii) $a \wedge b \leq a^*\&b$

for every $a, b \in R(Q)$.

PROOF. For any $a \in R(Q)$, we have

$$a\&a \leq a = a\&a^*\&a \leq a\&a,$$

which establishes (i). Moreover,

$$a = a\&a^*\&a \leq 1\&a^*\&a \leq a^*\&a,$$

since $a^*\&a \in L(Q)$. Thus, noting that the meet of right-sided elements is again right-sided, we have that

$$a \wedge b \leq (a \wedge b)^*\&(a \wedge b) \leq a^*\&b,$$

which establishes (ii).

COROLLARY. *If Q is a Gelfand quantale, then*

$$a \wedge a^{\perp} = 0$$

for all $a \in R(Q)$.

We note that the analogous results for $L(Q)$ are established similarly.

2. QUANTALES OF LINEAR RELATIONS

The aim now is to quantise the construction of the quantale $Q(X)$ of relations on a set X to yield the quantale $Q(H)$ of linear relations on a Hilbert space H. The quantale $Q(H)$ will be shown to be once again a Gelfand quantale, albeit with, as will also be seen to be the case with the quantale $Q(X)$, certain properties in addition which will be considered in the next section. The observation that allows this quantisation to take place is the following concerning the quantale $Q(X)$.

For any relation R on a set X, one may consider the mapping φ_R from the power set $\wp(X)$ of the set X to itself which assigns to each $A \in \wp(X)$ the subset

$$(A)\varphi_R = \{y \in X \mid \exists\, x \in A \ \ (x, y) \in R\}$$

of X. The mapping

$$\varphi_R : \wp(X) \to \wp(X)$$

is evidently a sup-preserving mapping from the sup-lattice $\wp(X)$ to itself. Moreover, it may be straightforwardly verified that any sup-preserving mapping

$$\varphi : \wp(X) \to \wp(X)$$

arises in this way from a unique relation on the set X. The multiplication of the quantale $Q(X)$ is exactly that corresponding to composition of mappings, with mappings composed in the categorical sense to the right of their arguments, and the supremum is that corresponding to the pointwise ordering of mappings into the sup-lattice $\wp(X)$. The unit of the quantale $Q(X)$, given by the equality relation on X, corresponds in this way to the identity mapping on $\wp(X)$.

The quantale $Q(X)$ may therefore be identified with the quantale of sup-preserving mappings from the sup-lattice $\wp(X)$ to itself. The way forward to the quantisation of these ideas is now evident. The sup-lattice $\wp(X)$ of subsets of the set X may be replaced by the sup-lattice $\wp(H)$ of closed linear subspaces of the Hilbert space H to yield the following definition:

Definition. By the *quantale* $\mathcal{Q}(H)$ *of linear relations* on a Hilbert space H will be meant the quantale of sup-preserving mappings from the sup-lattice $\wp(H)$ of closed linear subspaces of H to itself.

The sup-lattice $\wp(H)$ is canonically isomorphic to that of projections on the Hilbert space H. In particular, $\wp(H)$ is partially ordered by inclusion of closed linear subspaces, the supremum being given by taking the closure of the algebraic sum of the linear subspaces concerned. The quantale $\mathcal{Q}(H)$ has product given by composition of sup-preserving mappings and supremum defined pointwise with respect to the partial ordering on $\wp(H)$. Its unit is given by the identity operator.

Concerning this quantale, one may now prove the following:

Theorem 2. *For any Hilbert space H, the quantale* $\mathcal{Q}(H)$ *of linear relations on H is a Gelfand quantale.*

Before doing so, and without any comment on the semantics that we have in mind, we make the immediate observation that the quantale structure of $\mathcal{Q}(H)$ leads to the existence of a quantised analogue of the construction of the weakest prespecification of Hoare and He ([3]). Namely, given any elements $\varphi, \psi \in \mathcal{Q}(H)$, there is an element $\varphi\backslash\psi \in \mathcal{Q}(H)$ with the property that it is the largest element ξ of $\mathcal{Q}(H)$ satisfying the condition that

$$\xi \& \varphi \le \psi$$

in the quantale $\mathcal{Q}(H)$. Of course, the element is constructed exactly by taking the supremum of all elements of $\mathcal{Q}(H)$ satisfying this condition.

The sup-lattice $\wp(H)$ is rich in structure; in particular, it is *orthocomplemented*; that is, for every $M \in \wp(H)$ there exists a closed subspace $M^\perp$ of H such that

$$\begin{aligned} M^{\perp\perp} &= M \\ (\bigvee M_i)^\perp &= \bigwedge M_i^\perp \\ M \vee M^\perp &= H \\ M \wedge M^\perp &= 0\,. \end{aligned}$$

Using this structure, one can construct an operation * on $\mathcal{Q}(H)$ by defining

$$(M)\varphi^* = (\bigvee_{(N)\varphi \le M^\perp} N)^\perp$$

for all sup-preserving mappings from $\wp(H)$ to itself and for all $M, N \in \wp(H)$. In other words, $(M)\varphi^*$ is the closed subspace of H of which the orthogonal complement $((M)\varphi^*)^\perp$ is the largest satisfying

$$((M)\varphi^*)^\perp\varphi \le M^\perp.$$

If $N \le ((M)\varphi^*)^\perp$, then $(N)\varphi \le M^\perp$, so it is immediate that

$$N \le ((M)\varphi^*)^\perp \Leftrightarrow (N)\varphi \le M^\perp.$$

Thus, using the fact that $N \le M \Leftrightarrow M^\perp \le N^\perp$ we have that

$$\begin{aligned} N \le ((M)\varphi^{**})^\perp &\Leftrightarrow (N)\varphi^* \le M^\perp \\ &\Leftrightarrow M \le ((N)\varphi^*)^\perp \\ &\Leftrightarrow (M)\varphi \le N^\perp \\ &\Leftrightarrow N \le ((M)\varphi)^\perp, \end{aligned}$$

which shows that $(M)\varphi^{**} = (M)\varphi$ for all $M \in \wp(H)$, and, hence, that $\varphi^{**} = \varphi$.

A similar use of the properties of $\perp$ and of $*$ allows us to complete the verification of the fact that $*$ is an involution on $\mathcal{Q}(H)$, namely

$$\begin{aligned} (\varphi \& \psi)^* &= \psi^* \& \varphi^* \\ (\bigvee \varphi_i)^* &= \bigvee \varphi_i^*, \end{aligned}$$

for $\varphi, \varphi_i, \psi \in \mathcal{Q}(H)$.

In analysing the right-sided and left-sided elements of $\mathcal{Q}(H)$, we return to the analogy with the quantale $\mathcal{Q}(X)$ of relations on the set X, of which notion $\mathcal{Q}(H)$ is the quantisation. We have in that case described right-sided and left-sided relations as being of the form R_A and L_A, for $A \subseteq X$, where

$$R_A = \{(x,y) \in X \times X \mid x \in X - A\}$$

and

$$L_A = \{(x,y) \in X \times X \mid y \in A\}.$$

Under the identification of $\mathcal{Q}(X)$ with the quantale of sup-preserving mappings from $\wp(X)$ to itself, R_A and L_A clearly correspond, respectively, to the sup-preserving mappings λ_A and κ_A, where

$$(B)\lambda_A = \begin{cases} X & \text{unless} \\ \varnothing & B \subseteq A \end{cases}$$

and

$$(B)\kappa_A = \begin{cases} A & \text{unless} \\ \varnothing & B = \varnothing\,. \end{cases}$$

The obvious generalisation to $\mathcal{Q}(H)$ yields a description of the right- and left-sided elements. We define λ_M, for $M \in \wp(H)$, by

$$(N)\lambda_M = \begin{cases} H & \text{unless} \\ 0 & N \leq M, \end{cases}$$

for $N \in \wp(H)$. It is clear that λ_M is a sup-preserving mapping from $\wp(H)$ to itself and that

$$\lambda_M \& \varphi \leq \lambda_M$$

for all $\varphi \in Q(H)$. Moreover, if φ is any right-sided element of $Q(H)$, we must have

$$\varphi \& 1 \leq \varphi,$$

where

$$(N)1 = \begin{cases} H & \text{unless} \\ 0 & N = 0. \end{cases}$$

Then, clearly φ must be of the form λ_M, where

$$M = \bigvee \{ N \in \wp(H) \mid (N)\varphi = 0 \}.$$

Similarly, we define κ_M, for $M \in \wp(H)$, by

$$(N)\kappa_M = \begin{cases} M & \text{unless} \\ 0 & N = 0. \end{cases}$$

Sup-preserving mappings of the form κ_M are the left-sided elements of $Q(H)$.

As in the case of $Q(X)$, the verification that the involutive quantale $Q(H)$ is a Gelfand quantale, that is, that

$$\lambda_M \& \lambda_M^* \& \lambda_M = \lambda_M$$

for all $M \in \wp(H)$, turns on the fact that $\lambda_M^* = \kappa_{M^\perp}$. For, by definition,

$$(N)\lambda_M^* = (\bigvee_{(L)\lambda_M \leq N^\perp} L)^\perp.$$

Since $(L)\lambda_M \leq N^\perp$ when $N = 0$ and L is any closed linear subspace of H or when $N \neq 0$ and $L \leq M$, we see that

$$(0)\lambda_M^* = H^\perp = 0$$

and for $N \neq 0$,

$$(N)\lambda_M^* = M^\perp,$$

which shows that

$$\lambda_M^* = \kappa_{M^\perp}.$$

This completes the proof of the theorem, since it is easy to see that

$$\lambda_M \& \kappa_{M^\perp} \& \lambda_M = \lambda_M.$$

We remark that from the above relation,

$$\kappa_M^* = \lambda_{M^\perp}^{**} = \lambda_{M^\perp},$$

as well.

Finally, we examine the negation operations on $R(Q(H))$ and $L(Q(H))$ to establish their connection with the orthocomplement on $\wp(H)$ and the involutive structure of $Q(H)$. Recalling that for a right-sided element λ_A of $Q(H)$, $\lambda_A^\perp$ is the largest element such that

$$\lambda_A^* \& \lambda_A^\perp = 0,$$

we see that $\lambda_{A^\perp} \leq \lambda_A^\perp$ since clearly

$$\lambda_A^* \& \lambda_{A^\perp} = \kappa_{A^\perp} \& \lambda_{A^\perp} = 0.$$

On the other hand, if $\varphi \in R(Q(H))$ is such that

$$\kappa_{A^\perp} \& \varphi = 0,$$

it must be the case that $\varphi \leq \lambda_{A^\perp}$, so that

$$\lambda_A^\perp = \lambda_{A^\perp}.$$

A similar calculation shows that ${}^\perp\kappa_A = \kappa_{A^\perp}$ for any left-sided elemen κ_A of $Q(H)$. Putting these results together with those above, we see that the involution and the negation are related in the following way:

$$\lambda_A^* = {}^\perp\kappa_A$$
$$\kappa_A^* = \lambda_A^\perp.$$

3. Hilbert Quantales

It was seen in the last section that the construction of the quantale $Q(X)$ of relations on a set X could be quantised to yield a Gelfand quantale $Q(H)$ of linear relations on a Hilbert space H, thus permitting a quantised analogue of the weakest prespecification. The properties that were exploited in this quantisation were the identification of the quantale $Q(X)$ with the quantale of sup-preserving mappings from $\wp(X)$ to itself, and the structure of the sup-lattices $\wp(X)$ and $\wp(H)$. In particular, it was the fact that $\wp(H)$, the sup-lattice of closed linear subspaces of the Hilbert space H, is orthocomplemented that proved to be of great significance, replacing the complementation present in the sup-lattice $\wp(X)$ of subsets of the set X.

We now extract these salient facts and generalise the construction of the quantale of linear relations to a more abstract setting in which we begin by being provided with an orthocomplemented sup-lattice S. We recall that a sup-lattice S is said to be *orthocomplemented* if there exists a unary operation $\perp$ on S satisfying the conditions that:

$$\begin{aligned} s^{\perp\perp} &= s \\ (\bigvee s_i)^{\perp} &= \bigwedge s_i^{\perp} \\ s \vee s^{\perp} &= 1 \\ s \wedge s^{\perp} &= 0\,, \end{aligned}$$

for all $s, s_i \in S$.

DEFINITION. By the *quantale* $Q(S)$ *of endomorphisms* of the orthocomplemented sup-lattice S will be meant the unital quantale of sup-preserving mappings from S to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication corresponding to composition of mappings, and with the unit given by the identity mapping.

As in the case of $Q(H)$, we can define an involution $*$ on $Q(S)$ by setting

$$(s)\varphi^* = (\bigvee_{(t)\varphi \le s^{\perp}} t)^{\perp},$$

for $\varphi \in Q(S)$ and $s, t \in S$; that is, $(s)\varphi^*$ is the element of S of which the orthogonal complement is the largest element satisfying

$$((s)\varphi^*)^{\perp}\varphi \le s^{\perp}.$$

Using the fundamental property

$$t \le ((s)\varphi^*)^{\perp} \Leftrightarrow (t)\varphi \le s^{\perp},$$

which is true for all $\varphi \in Q(S)$ and $s, t \in S$, one can verify that $Q(S)$ is an involutive quantale with $*$ as its involution, which is the first step towards establishing:

THEOREM 3. *For any orthocomplemented sup-lattice S, the quantale* $Q(S)$ *of sup-preserving mappings from S to itself is a Gelfand quantale.*

The right- and left-sided elements of $Q(S)$ can be described in a manner totally analogous to those of $Q(H)$. We define for $s \in S$, the sup-preserving mappings λ_s and κ_s, given by:

$$(t)\lambda_s = \begin{cases} 1 & \text{unless} \\ 0 & t \le s \end{cases}$$

and

$$(t)\kappa_s = \begin{cases} s & \text{unless} \\ 0 & t = 0\,. \end{cases}$$

All right-sided elements of $Q(S)$ are of the form λ_s for some $s \in S$, whilst mappings of the form κ_s characterise the left-sided elements.

The same properties relating λ, κ, the involution, and the negation operation available on $R(Q(S))$ and $L(Q(S))$, as defined in Section 1, also pertain:

$$\begin{aligned} \lambda_s^* &= \kappa_{s^{\perp}} \\ \lambda_s^{\perp} &= \lambda_{s^{\perp}} \\ {}^{\perp}\kappa_s &= \kappa_{s^{\perp}}. \end{aligned}$$

These in turn imply that:

$$\begin{aligned} \kappa_s^* &= \lambda_{s^{\perp}} \\ \kappa_s^* &= {}^{\perp}\kappa_s \\ \kappa_s^* &= \lambda_s^{\perp}. \end{aligned}$$

The final step in the proof of the theorem, namely that

$$\lambda_s \& \lambda_s^* \& \lambda_s = \lambda_s,$$

follows readily from these relations.

Since these quantales of the form $Q(S)$ have arisen in considering generalisations of the quantale $Q(H)$ of linear relations on a Hilbert space H, we make the following definition:

DEFINITION. A Gelfand quantale Q will be said to be a *Hilbert quantale* if it is isomorphic to a Gelfand quantale of the form $Q(S)$ for some orthocomplemented sup-lattice S.

Our aim is now to characterise such quantales.

We recall from the Proposition and Corollary of Section 1 that whenever Q is a Gelfand quantale, then the quantale $R(Q)$ of right-sided elements of Q is an idempotent right-sided quantale for which the negation operation $\perp$, given by

$$a^{\perp} = \bigvee_{a^* \& b = 0} b,$$

satisfies

$$\begin{aligned} a &\le a^{\perp\perp}, \\ (\bigvee a_i)^{\perp} &= \bigwedge a_i^{\perp}, \end{aligned}$$

and

$$a \wedge a^{\perp} = 0,$$

for each $a, a_i \in R(Q)$, but which is not necessarily an orthocomplement. On the other hand, if Q is of the form $Q(S)$ for some orthocomplemented sup-lattice S, then the negation $\perp$ on $R(Q(S))$ is necessarily an orthocomplement in view of:

PROPOSITION. *For any orthocomplemented sup-lattice S, the mapping*

$$\chi : S \to R\,(Q\,(S))$$

defined by

$$(s)\chi = \lambda_{s^\perp}$$

is an isomorphism of orthocomplemented sup-lattices.

PROOF. The mapping χ is clearly onto. It is one-one since if $s \leq t$ and $s \neq t$, then

$$(s^\perp)\lambda_{s^\perp} = 0, \quad (s^\perp)\lambda_{t^\perp} = 1$$

and if $s \not\leq t$, then

$$(t^\perp)\lambda_{s^\perp} = 1, \quad (t^\perp)\lambda_{t^\perp} = 0.$$

The mapping χ preserves order since

$$s \leq t \Rightarrow t^\perp \leq s^\perp \Rightarrow \lambda_{s^\perp} \leq \lambda_{t^\perp} \Rightarrow (s)\chi \leq (t)\chi.$$

Finally, we note that

$$(s^\perp)\chi = \lambda_{s^{\perp\perp}} = (\lambda_{s^\perp})^\perp = (s)\chi^\perp.$$

COROLLARY. *For any orthocomplemented sup-lattice S, there is an isomorphism*

$$\sigma : Q\,(S) \to Q\,(R(Q\,(S)))$$

of Gelfand quantales given by

$$(\varphi)\sigma = \chi^{-1}\varphi\chi$$

for each $\varphi \in Q\,(S)$.

In the general case of an arbitrary Gelfand quantale Q, however, there already exists a mapping

$$\mu : Q \to Q\,(R(Q))$$

given by $(a)\mu = \pi_a$, where

$$(b)\pi_a = a^*\&b,$$

which clearly preserves sups, &, and the unit. If we further assume that $R(Q)$ is orthocomplemented, then for μ to be a homomorphism of Gelfand quantales it is necessary and sufficient to verify that

$$(a^*)\mu = ((a)\mu)^*$$

for all $a \in Q$. If $b \in Q$, we have

$$(b)(a^*)\mu = (b)\pi_{a^*} = a^{**}\&b = a\&b$$

and

$$\begin{aligned} (b)((a)\mu)^* &= (b)\pi_a^* = (\textstyle\bigvee_{(c)\pi_a \leq b^\perp} c)^\perp = (\bigvee_{a^*\&c \leq b^\perp} c)^\perp \\ &= (\textstyle\bigvee_{b^*\&a^*\&c=0} c)^\perp = (\bigvee_{(a\&b)^*\&c=0} c)^\perp = (a\&b)^{\perp\perp} \end{aligned}$$

What is proved is:

PROPOSITION. *For any Gelfand quantale Q,*

$$\mu : Q \to Q\,(R(Q))$$

is a homomorphism of Gelfand quantales if, and only if, $R(Q)$ *is orthocomplemented.*

Thus, when Q is a Hilbert quantale, that is, a quantale isomorphic to one of the form $Q\,(S)$ for some orthocomplemented sup-lattice S, the situation at which we have arrived is that there exist two mappings:

$$\sigma, \mu : Q \to Q\,(R(Q))\,,$$

one of which has been shown to be an isomorphism. We claim that σ and μ are the same in this case.

To wit, identifying Q with the quantale $Q\,(S)$ to which it is isomorphic, let $\varphi \in Q\,(S)$ ($= Q$) and consider an arbitrary element λ_s of $R(Q)$. We have

$$\begin{aligned} (\lambda_s)(\varphi)\sigma &= (\lambda_s)\chi^{-1}\varphi\chi = (s^\perp)\varphi\chi \\ &= \lambda_{((s^\perp)\varphi)^\perp} \\ (\lambda_s)(\varphi)\mu &= (\lambda_s)\pi_\varphi = \varphi^*\&\lambda_s\,. \end{aligned}$$

For $t \in S$,

$$(t)\lambda_{((s^\perp)\varphi)^\perp} = \begin{cases} 1 & \text{unless} \\ 0 & t \leq ((s^\perp)\varphi)^\perp \end{cases}$$

and

$$(t)\varphi^*\&\lambda_s = \begin{cases} 1 & \text{unless} \\ 0 & (t)\varphi^* \leq s\,. \end{cases}$$

But $(t)\varphi^* \leq s \Leftrightarrow t \leq ((s^\perp)\varphi)^\perp$ by the fundamental property of the involution on $Q\,(S)$.

Recalling that the canonical morphism $\mu : Q \to Q\,(R(Q))$ is a homomorphism of Gelfand quantales if, and only if, $R(Q)$ is orthocomplemented, we have achieved the following characterisation of Hilbert quantales:

Theorem 4. *A Gelfand quantale Q is a Hilbert quantale if, and only if, the canonical mapping*

$$\mu : Q \to Q\,(R(Q))$$

is an isomorphism of Gelfand quantales.

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