Quantales: Quantal Sets

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Introduction

In this paper, we investigate the ideas involved in extending to the non-commutative case the concepts of presheaf and sheaf on a locale. The context within which we shall work will be that of quantales, introduced by the first author [10,11] to provide a framework for the development of the Gelfand representation of not necessarily commutative C*-algebras, extending the lattice theoretic ideas of Dilworth and Ward [3], and the spectral insights of Giles and Kummer [7] and of Akemann [1]. The ideas which we present are largely contained in the thesis of the second author [15], although their development here will be from a very slightly different perspective. That this has taken a little time to bring to print is due in part to the publication of a summary of the thesis elsewhere [2], developing the ideas further in another direction. An alternative approach to the problem discussed there will be undertaken in a sequel to the present paper [14].

The particular quantales that will be considered here arise, amongst other places, in the context of considering the spectrum Max A of a C*-algebra A. At the time that the work was done, the quantales which will be considered here, which we now refer to as *right Gelfand quantales*, were being actively investigated in this context, following the work of Giles and Kummer [7] on spectral representations of C*-algebras. Subsequently, due to the insightful contributions of Rosicky [17], a broader context, namely that of Gelfand quantales, was found for these ideas [12]. These right Gelfand quantales were then seen to arise typically as the right sides of Gelfand quantales, with this same right side carrying the topological structure of the spectrum within the ambient structure of the Gelfand quantale [13].

Although the concept of sheaf which one might now wish to define would be one that is defined on the Gelfand quantale itself, the present study of a concept defined on a right Gelfand quantale would be expected to form the right reflection, in some sense, of that more general concept yet to be studied in any depth. The beginnings of approaches to this may be found in the papers of Nawaz [16] and of French [5]. A definitive attack on these ideas, with the aim of obtaining a sheaf theoretic context for the Gelfand representation of non-commutative C*-algebras, remains to be mounted. Our belief, however, is that the

concepts of quantal set and of sheaf arising in that more general context will inevitably admit a reflection to the right (and left) sides of the Gelfand quantale Max A concerned. For the moment, therefore, we investigate only the case of a right Gelfand quantale, both for its own interest, and to provide a point of reference for the development of the more general theory.

Within this context, in which many of the difficulties involved in working over a quantale are overcome due to the particular properties encountered, it proves possible to arrive at appropriate definitions of the concepts of presheaf and of sheaf, and to show the existence of a reflection of the category of presheaves into the category of sheaves on the quantale, corresponding to the process of sheafification. The approach taken to achieving this, however, is from the concept of a quantal set, generalising that of Heyting-valued, or as we would prefer to call it, local, set, introduced initially by Higgs [8] as a generalisation, and extension, of the notion of a Boolean-valued set [18], and further developed by Fourman and Scott [6].

The concept of quantal set is fundamental to the discussion, the principle being that a definition of the category of quantal sets over a right Gelfand quantale Q is already an exacting task within this non-commutative context. To show that the concept of completeness may be extended to this situation itself provides some indication that the definitions taken are in some measure correct. While to find that these ideas may be tied consistently in to concepts of presheaf, and of sheaf, in a way that directly extends the localic case, while introducing aspects that reflect the non-commutative context within which one is working, gives an impression that the concepts introduced are, in some sense or other, quite natural.

The agenda, in some sense, is that one knows in advance the theorems that are to be proved. The question is whether one can find the definitions that allow their proof. The constraints are that, certainly, the concepts should become those with which one is familiar in the case that the quantale concerned is actually a locale, while, in the case that it is not, the quantale Q itself should still be able to be recovered from the quantal sets defined over it, by considering the subsets of the quantal set $\mathbb{1}_Q$. Indeed, these ideas should extend to the interpretation of a non-commutative predicate calculus in which the structure of quantal sets is reflected in the existence of quantal sets of subsets based on exponentiation of a quantal subset classifier [14]. Furthermore, as a subtext to this agenda, one has the requirement that the concepts should appear natural within, indeed should reflect, the non-commutative context within which the theory is developed.

The development of the logical aspects of these ideas gives further indication that the structures concerned are of intrinsic interest, although these aspects are those left to a later paper. The ideas involved will be alluded to in the final section of the present paper, in which a particularly murky aspect of the theory developed in the earlier sections is finally revealed. In those earlier sections, it has been our decision to include more detail in the proofs than might normally be considered necessary. This is in part due to our own experience of finding that the manipulations needed to establish a particular inequality were often more easily lost than found. Having found them, it seemed wise to write them down.

1. Quantales

The concept of a quantale was introduced [10,11] to provide a non-commutative generalisation of that of a locale [9]. On the one hand, this generalisation was intended to allow the consideration on non-commutative spaces, in particular that of the spectrum of a C*-algebra. On the other, the concept was expected to relate to the semantics of non-commutative logics, in particular that of quantum mechanics. The formalisation of these ideas leads to the following:

DEFINITION. By a *quantale* Q will be meant a lattice having arbitrary joins $\bigvee$, together with an associative product & satisfying

$$a \,\&\, \bigvee b_i = \bigvee a\&b_i \text{ and } \bigvee a_i \,\&\, b = \bigvee a_i\&b$$

for all $a, b, a_i, b_i \in Q$.

It may be remarked that any quantale Q has a greatest element, 1_Q, and a least element, 0_Q, which will be referred to respectively as the *identity* and the *zero* elements of the quantale. As one consequence of the distributivity of the product & over any arbitrary join $\bigvee$, it may be remarked that in any quantale Q one has that

$$p \leq q \quad \text{implies that} \quad p \,\&\, r \leq q \,\&\, r$$

for any $p, q, r \in Q$, with a similar result holding when taking the product on the left. One also has that

$$0_Q \,\&\, q = 0_Q = q \,\&\, 0_Q$$

for any $q \in Q$.

Any locale L is a quantale, by taking the product & to be the operation of meet $\wedge$ in the lattice L. Since the identity 1_L in this case satisfies the condition that

$$1_Q \wedge q = q = q \wedge 1_Q$$

for any $q \in L$, the identity of a locale is actually a unit for the quantale, in the following sense:

DEFINITION. By a *unit* $e \in Q$ for a quantale Q will be meant an element of Q satisfying

$$e \,\&\, q = q = q \,\&\, e$$

for all $q \in Q$. A quantale Q will be said to be *unital* provided that it admits a unit, necessarily unique.

Although in any locale L the identity 1_L is a unit for the locale, in a quantale that is unital this is generally far from being the case. Indeed, a more typical example of a quantale that is unital is given by the following [13]:

DEFINITION. By the *spectrum* Max A of a C*-algebra A will be meant the quantale of closed linear subspaces of A, together with the join $\bigvee$ given by

$$\bigvee_i M_i = \overline{\Sigma_i M_i}$$

for any $M_i \in \text{Max } A$, and the product & defined by

$$M \,\&\, N = \overline{M \cdot N}$$

for any $M, N \in \text{Max } A$.

The product & is given by taking the closure of the algebraic product

$$M \cdot N = \{ \Sigma_i a_i b_i \mid a_i \in M, b_i \in N \}$$

of the closed linear subspaces of A, while the join $\bigvee$ is given by taking the closure of the algebraic sum

$$\Sigma_i M_i = \{ \Sigma_j a_{i_j} \in A \mid a_{i_1} \in M_{i_1}, \ldots, a_{i_n} \in M_{i_n} \}.$$

It is evident that the closed linear subspace $e \in \text{Max } A$ generated by the unit $1 \in A$ of the C*-algebra A, always assumed to be unital, is a unit for the quantale Max A. It may also be remarked that this is certainly not the identity of the quantale, which is the C*-algebra A itself.

Although the identity of a quantale Q is generally not a unit for the quantale, it is of interest to consider those elements on which it acts on one side or another as a unit for the product, in the sense of the following:

DEFINITION. An element $q \in Q$ of a quantale Q will be said to be *right-sided* (respectively *left-sided*) in the case that

$$q \,\&\, 1_Q \leq q \text{ (respectively } 1_Q \,\&\, q \leq q \text{)}.$$

An element $q \in Q$ that is both right-sided and left-sided will be said to be *two-sided*.

It may be remarked immediately that in any quantale Q that is unital, the inequalities may be replaced by equalities, since

$$q = q \,\&\, e \leq q \,\&\, 1_Q \quad \text{(respectively } q = e \,\&\, q \leq 1_Q \,\&\, q \text{)}$$

for any $q \in Q$.

We have already noted that in a locale L, the identity is a unit for the quantale, hence every element of the locale is two-sided. However, in the quantale Max A that is the spectrum of a C*-algebra A, the right-sided (respectively left-sided) elements are exactly the closed right (respectively left) ideals of A, since, for any $M \in \text{Max } A$, the product

$$M \,\&\, A \text{ (respectively } A \,\&\, M \text{)}$$

is exactly the closure of the right (respectively left) ideal generated by the closed linear subspace M. In particular, the two-sided elements of Max A are the closed ideals of the C*-algebra A.

For any quantale Q, the right-sided (respectively left-sided) elements again form a quantale with respect to the operations of the quantale Q, which we shall denote by

$$R(Q) \text{ (respectively } L(Q) \text{)},$$

referred to as the *right* (respectively the *left*) *side* of the quantale Q. Similarly, the two-sided elements are closed under product and arbitrary joins, hence form a quantale

$$I(Q),$$

the *two-sided part* of the quantale Q. It should, however, be noted that the right sided part $R(Q)$ of a unital quantale Q may in general no longer be unital. Indeed, if it has a unit, then this is necessarily the identity of the quantale Q, from which it follows that

$$R(Q) = I(Q).$$

It may be remarked that for the spectrum Max A of a C*-algebra A, the quantale

$$I(\mathrm{Max}\ A)$$

is actually a locale. Moreover, for a commutative C*-algebra A, this locale is the spectrum of A in the conventional sense that the locale of closed ideals of A is that of the weak* topology on the space of multiplicative linear functionals on A. Concerning the quantale

$$R(\mathrm{Max}\ A)$$

(with corresponding assertions about the quantale $L(\mathrm{Max}\ A)$), it may be shown that

$$I \,\&\, I = I$$

for every closed right ideal $I \in R(\mathrm{Max}\ A)$, a result which is a consequence [15] of the existence of approximate units in arbitrary C*-algebras [4], and which leads us to the following:

DEFINITION. A quantale Q will be said to be *idempotent* provided that

$$p \,\&\, p = p$$

for each $p \in Q$, and to be *right-sided* provided that

$$p \,\&\, 1_Q = p$$

for each $p \in Q$.

Any locale L is an idempotent right-sided quantale that is also unital. Moreover, conversely, any idempotent, right-sided quantale Q that is unital has its product given by the meet of the lattice Q, hence is actually a locale. For, one observes firstly that the unit $e \in Q$ of such a quantale Q must actually be the identity of Q, since $e \,\&\, 1_Q = e$ by the right-sidedness of the quantale, while $e \,\&\, 1_Q = 1_Q$ since $e \in Q$ is a unit. Hence, any element $p \in Q$ is necessarily two-sided. Thus, given any elements $p, q \in Q$ of the quantale Q, we have, on the one hand, that

$$p \,\&\, q \le p \,\&\, 1_Q = p \quad \text{and} \quad p \,\&\, q \le 1_Q \,\&\, q = q,$$

which implies that $p \,\&\, q \le p \wedge q$, while, on the other,

$$p \wedge q \le p \quad \text{and} \quad p \wedge q \le q,$$

which implies that $p \wedge q = (p \wedge q) \,\&\, (p \wedge q) \le p \,\&\, q$, yielding the required equality. Since the product, now known to be the meet, distributes over arbitrary joins, the quantale Q is a locale.

For an idempotent, right-sided quantale Q which is not necessarily unital, there is still a significant way that one may go along the path towards locales, in the following [15]:

THEOREM. *For any idempotent, right-sided quantale Q, one has that*

$$p \,\&\, q \,\&\, r = p \,\&\, r \,\&\, q$$

for any elements $p, q, r \in Q$. In particular, for any element $p \in Q$, the quantale

$$Q_p = \{\, p \,\&\, q \in Q \mid q \in Q \,\}$$

obtained by localising at the element $p \in Q$ is a locale.

Although its consequences are extensive, the condition asserted, which we shall refer to as the *right-symmetricity* of the quantale Q, is almost trivially proved. For any $p, q, r \in Q$, one has that

$$\begin{aligned} p \,\&\, q \,\&\, r &= p \,\&\, q \,\&\, r \,\&\, p \,\&\, q \,\&\, r \\ &\le p \,\&\, 1_Q \,\&\, r \,\&\, 1_Q \,\&\, q \,\&\, 1_Q \\ &= p \,\&\, r \,\&\, q, \end{aligned}$$

by respectively the idempotency of Q, the monotonicity of the product of Q, and the right-sidedness of the elements of Q. Similarly,

$$p \,\&\, r \,\&\, q \le p \,\&\, q \,\&\, r,$$

giving the required result.

Considering now the subset Q_p, we observe that, by the distributivity of the product over the arbitrary joins in the quantale, the subset is closed under arbitrary joins; while, by the idempotency and the right-symmetricity of the quantale, one has that

$$(p \,\&\, q) \,\&\, (p \,\&\, r) = p \,\&\, p \,\&\, q \,\&\, r = p \,\&\, (q \,\&\, r),$$

giving that it is closed under the product. The quantale is idempotent, since by the above identity, one has that

$$(p \,\&\, q) \,\&\, (p \,\&\, q) = p \,\&\, (q \,\&\, q) = p \,\&\, q$$

It is also right-sided: for $p = p \,\&\, 1_Q \in Q_p$, by the right-sidedness of Q. Hence, $p \in Q_p$ is the identity of the quantale Q_p, since $p \,\&\, q \le p \,\&\, 1_Q = p$ for any $q \in Q$. But then,

$$(p \,\&\, q) \,\&\, p = p \,\&\, p \,\&\, q = p \,\&\, q.$$

But, by a similar argument on the left side, we have that

$$p \,\&\, (p \,\&\, q) = (p \,\&\, p) \,\&\, q = p \,\&\, q,$$

giving that $p \in Q_p$ is actually a unit for the quantale Q_p. Thus, by our earlier remarks, the quantale Q_p is actually a locale. ∎

There is therefore a sense in which an idempotent, right-sided quantale Q may be considered as being constructed from the locales Q_p obtained by localising at each $p \in Q$, with the quantale Q itself binding these locales together in a non-commutative manner. An immediate corollary to this result is that obtained by localising at the identity element of the quantale, extending the observation already made in the case of the spectrum Max A of a C*-algebra A:

COROLLARY. *For any idempotent, right-sided quantale Q, the quantale*

$$I(Q)$$

of two-sided elements of Q is necessarily a locale. Moreover, the idempotent, right-sided quantale Q is a locale exactly if Q is unital. ∎

It may be observed in passing that in any idempotent quantale Q, one again has that an element is right-sided (respectively left-sided) exactly if

$$q \& 1_Q = q \quad (\text{respectively} \quad 1_Q \& q = q)$$

for any $q \in Q$, since already we have that

$$q = q \& q \leq q \& 1_Q \quad (\text{respectively} \quad q = q \& q \leq 1_Q \& q)$$

In the case of the spectrum Max A of a C*-algebra A, it is the idempotent, right-sided quantale

$$R(\text{Max } A),$$

or equivalently the involutively isomorphic idempotent, left-sided quantale $L(\text{Max } A)$, that appears to carry the topological structure of the spectrum [13]. Indeed, it was these lattices of closed right and left ideals that were identified by Giles and Kummer [7], and by Akemann [1], as providing the non-commutative topology in terms of which the Gelfand representation of the C*-algebra might be expressed.

The development of these ideas into the context of quantales, and particularly the introduction of a concept of complete regularity within this context, and the observation that this could be defined intrinsically within the framework of Gelfand quantales, provide incentive for investigating the concepts of quantal set, and of presheaf and sheaf, over quantales that are idempotent and right-sided.

In view of the connection with Gelfand quantales [12], on the one hand, and with the Gelfand representation of C*-algebras [13], on the other, we shall make the following:

DEFINITION. By a *right Gelfand* quantale Q will be meant a quantale that is idempotent, and right-sided.

It is with these quantales that we shall be concerned for the rest of this paper, and the quantales considered should throughout be assumed to be right Gelfand quantales unless otherwise explicitly stated.

2. QUANTAL SETS

Consider now a quantale Q, henceforth always assumed to be idempotent and right-sided, that is to say, a right Gelfand quantale. It will be recalled that such a quantale Q is a locale exactly if it is unital. Our concern now is to describe a concept of quantal set over the quantale, in the sense of a set together with extent and equality relations taking values in Q, in such a way that this coincides with that of a Heyting-valued set over Q in the case that the quantale is actually a locale [8], yet which reflects its non-commutativity in the case that it is not.

DEFINITION. Given a right Gelfand quantale Q, by a *quantal set* over Q will be meant a set A, together with mappings:

$$E : A \to Q$$

and $$[\![\cdot = \cdot]\!] : A \times A \to Q,$$

which together satisfy the following conditions:

i) $Ea \& Ea \leq [\![a = a]\!]$;

ii) $[\![a = b]\!] \leq Ea \& Eb$;

iii) $Ea \& [\![b = a]\!] \leq [\![a = b]\!]$;

iv) $[\![a = b]\!] \& [\![b = c]\!] \leq [\![a = c]\!]$,

for all $a, b, c \in A$.

The elements Ea and $[\![a = b]\!]$ of the quantale Q will be called respectively the *extent* and the *equality* of the quantal set A.

It may be observed immediately that:

$$Ea = [\![a = a]\!]$$

for each $a \in A$, because of the idempotency of the quantale Q. It will be noted that the equality of a quantal set is not required to be symmetric, but rather to satisfy a condition akin to that of right symmetricity satisfied by the quantale. It will be seen that this condition implies that the equality is actually symmetric in the case that the quantale is indeed a locale.

For any quantale Q, it may be deduced from its right symmetricity that the equality of a quantal set satisfies the condition that:

$$Ea \& [\![a = b]\!] = [\![a = b]\!] = [\![a = b]\!] \& Eb$$

for all $a, b \in A$. From this observation it follows that the equality of the quantal set is right symmetric, in the sense that:

$$p\&[\![a = b]\!] = p\&[\![b = a]\!],$$

for any $p \in Q$ and for all $a, b \in A$. That the equality is not generally symmetric can be seen from the following:

EXAMPLE. Given a quantale Q, consider the quantal set

$$\mathbb{1}_Q$$

obtained by taking the set Q, together with the extent and equality relations defined by:

$$Eq = q,$$

and $$[\![p = q]\!] = p\&q,$$

for all $p, q \in Q$.

It may be observed that in showing that this is a quantal set only the assumption that the quantale is right-sided is needed. The equality evidently is symmetric exactly if the quantale is commutative. In the presence of the assumption of idempotency also, this is again equivalent to the quantale being a locale, since the identity is then a unit for the quantale.

Now the idea of a map of quantal sets over the quantale Q may be introduced by the following:

DEFINITION. Given a right Gelfand quantale Q, by a *map of quantal sets*

$$f : A \to B$$

over the quantale Q will be meant a mapping

$$f : A \times B \to Q$$

which satisfies the following conditions:

i) $f(a, b) \le Ea\&Eb$;

ii) $[\![a = a']\!]\&f(a', b) \le f(a, b)$;

iii) $f(a, b)\&[\![b = b']\!] \le f(a, b')$;

iv) $Eb\&Eb'\&f(a, b)\&f(a, b') \le [\![b = b']\!]$;

v) $Ea \le \bigvee_b f(a, b)$,

for all $a, a' \in A$ and $b, b' \in B$.

One is therefore describing a relation defined on $A \times B$ over the quantale Q, which is required to be functional with respect to the quantale Q. Again, the existence of a right identity element in the quantale allows it to be shown that:

$$Ea = \bigvee_b f(a, b),$$

for each $a \in A$. Moreover, from the right symmetricity of the quantale, one deduces also that:

$$Ea\&f(a, b) = f(a, b) = f(a, b)\&Eb\ ,$$

for all $a \in A$ and $b \in B$.

EXAMPLE. For any quantal set A over Q, there is a map:

$$\tau_A : A \to \mathbb{1}_Q$$

of quantal sets, defined by assigning to each $a \in A$ and each $q \in Q$ the element:

$$\tau_A(a, q) = Ea\&q$$

of the quantale Q.

It may be seen straightforwardly that this is a map of quantal sets. For instance, to verify that

$$[\![a = a']\!]\&\tau_A(a', q) \le \tau_A(a, q),$$

we note that $[\![a = a']\!] \le Ea\&Ea'$, hence, $[\![a = a']\!]\&Ea' \le Ea\&Ea' \le Ea$, and so, $[\![a = a']\!]\&Ea'\&q \le Ea\&q$. It will be seen later that this map is the unique map of quantal sets from A to the quantal set $\mathbb{1}_Q$.

With these preliminaries, we may now state the following:

THEOREM. *For any right Gelfand quantale Q, the quantal sets and the maps of quantal sets over Q determine a category*

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of quantal sets over the quantale Q.

Evidently, it remains to define the notions of the identity map on any quantal set, and of the composition of maps of quantal sets: firstly, for any quantal set A, define

$$id_A : A \to A$$

to be given by the equality relation

$$[\![\cdot\ =\ \cdot]\!] : A \times A \to Q\ ;$$

next, for any maps $f : A \to B$, $g : B \to C$ of quantal sets, define

$$gf : A \to C$$

to be the relation given by setting

$$(gf)(a,c) = \bigvee_b f(a,b)\&g(b,c)$$

for each $a \in A$ and $c \in C$.

Then it may be verified straightforwardly that these yield a category of quantal sets and maps of quantal sets over Q, although it may be remarked that the functionality of the composite of maps depends critically on the right symmetricity of the quantale. Explicitly, to verify that

$$Ec\&Ec'\&h(a,c)\&h(a,c') \leq [\![c = c']\!]$$

for the composite h of f and g, one observes that the left hand side is given by:

$$Ec\&Ec'\& \bigvee_b f(a,b)\&g(b,c)\& \bigvee_{b'} f(a,b')\&g(b',c'),$$

which by distributivity may be written

$$\bigvee_{b,b'} Ec\&Ec'\&f(a,b)\&g(b,c)\&f(a,b')\&g(b',c').$$

Applying the right symmetricity of the quantale, and the functionality of f and of g, one obtains successively:

$$= \bigvee_{b,b'} Ec\&Ec'\&f(a,b)\&f(a,b')\&g(b,c)\&g(b',c')$$

$$= \bigvee_{b,b'} Ec\&Ec'\&Eb\&Eb'\&f(a,b)\&f(a,b')\&g(b,c)\&g(b',c')$$

$$\leq \bigvee_{b,b'} Ec\&Ec'\&[\![b = b']\!]\&g(b,c)\&g(b',c')$$

$$= \bigvee_{b,b'} Ec\&Ec'\&g(b,c)\&[\![b = b']\!]\&g(b',c')$$

$$\leq [\![c = c']\!],$$

which establishes the functionality of the composite. That the identity maps of the category compose identically follows from the extensionality of any map of quantal sets, together with the observation made earlier that:

$$Ea\&f(a,b) = f(a,b) = f(a,b)\&Eb\ ,$$

for any $a \in A$ and $b \in B$. All other details of the proof will be omitted. ▯

The immediate observation that one may make is then:

Corollary. *For any locale Q, the category*

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is exactly the category of local sets and of maps of local sets over the locale Q.

It has already been remarked that in this case the commutativity of the multiplication of the quantale yields that equality is necessarily symmetric on any quantal set: for,

$$[\![a = b]\!] = [\![a = b]\!]\&Eb = Eb\&[\![a = b]\!]$$

implies that $[\![a = b]\!] \leq [\![b = a]\!]$, by the condition which gives the right symmetricity of equality, from which it follows that:

$$[\![a = b]\!] = [\![b = a]\!],$$

for all $a, b \in A$, by interchanging the elements considered. But, conversely, any local set over the locale has an equality which *a fortiori* is right symmetric. Hence, the concept of quantal set coincides with that of local set, in the case of a locale Q.

Equally, the conditions which define a map of quantal sets are equivalent to those for a map of local sets over the locale Q. Only that expressing functionality needs the observation that:

$$Eb\&Eb'\&f(a,b)\&f(a,b') = f(a,b)\&Eb\&f(a,b')\&Eb' = f(a,b)\&f(a,b'),$$

in this case, which yields that expressing the functionality of a map of local sets over the locale Q. ▯

More generally, one may obtain the following characterisation of isomorphisms in the category of quantal sets over any quantale:

Corollary. *For any right Gelfand quantale Q, a map*

$$f : A \to B$$

of quantal sets over Q is an isomorphism if, and only if, the following conditions are satisfied:

i) $f(a,b)\&f(a',b) \leq [\![a = a']\!]$

for all $a, a' \in A$ *and any* $b \in B$ *;*

and ii) $Eb \leq \bigvee_a Eb\&f(a,b)$

for all $b \in B$.

For suppose that

$$f : A \to B$$

is an isomorphism of quantal sets over Q, of which the inverse is

$$g : B \to A.$$

Then, firstly, $\bigvee_b f(a,b)\&g(b,a') = [\![a = a']\!]$ for all $a, a' \in A$, because this composite is the identity map on the quantal set A. Hence,

$$f(a,b)\&g(b,a') \leq [\![a = a']\!]$$

for each $b \in B$, and

$$Ea' = \bigvee_{b'} f(a',b')\&g(b',a')$$

for all $a' \in A$. Then,

$$\begin{aligned} f(a,b)\&f(a',b) &= f(a,b)\&Ea'\&f(a',b) = f(a,b)\&f(a',b)\&Ea' \\ &= \bigvee_{b'} f(a,b)\&f(a',b)\&f(a',b')\&g(b',a') \\ &\leq \bigvee_{b'} f(a,b)\&[\![b=b']\!]\&g(b',a') \leq f(a,b)\&g(b,a') \\ &\leq [\![a=a']\!]. \end{aligned}$$

Now, secondly, the fact that composition in the reverse order yields the identity map on the quantal set B gives that

$$[\![b=b']\!] = \bigvee_a g(b,a)\&f(a,b')$$

for each $b, b' \in B$, hence, in particular, that

$$Eb = \bigvee_a g(b,a)\&f(a,b) = \bigvee_a Eb\&g(b,a)\&f(a,b) \leq \bigvee_a Eb\&f(a,b).$$

So, both conditions are therefore satisfied.

Conversely, given a map of quantal sets

$$f : A \to B$$

satisfying these conditions, consider the mapping

$$g : B \times A \to Q$$

defined by

$$g(b,a) = Eb\&f(a,b)$$

for each $a \in A$ and $b \in B$. It is asserted, firstly, that this determines a map of quantal sets

$$g : B \to A,$$

and, moreover, that this provides an inverse for the given map. For, the extensionality of the relation defined may be straightforwardly verified. That it is functional and total is seen by:

$$\begin{aligned} Ea\&Ea'\&g(b,a)\&g(b,a') &= Ea\&Ea'\&Eb\&f(a,b)\&Eb\&f(a',b) \\ &\leq Ea\&Ea'\&f(a,b)\&f(a,'b) \leq Ea\&Ea'\&[\![a=a']\!] \\ &= Ea\&[\![a=a']\!]\&Ea' \\ &= [\![a=a']\!], \end{aligned}$$

by applying the first condition; while, by the second, one has that

$$\begin{aligned} &[\![b=b']\!] = Eb\&[\![b=b']\!] \leq \bigvee_a Eb\&f(a,b)\&[\![b=b']\!] \\ &\leq \bigvee_a Eb\&f(a,b)\&f(a,b)\&[\![b=b']\!] = \bigvee_a g(b,a)\&f(a,b'), \end{aligned}$$

while, conversely,

$$\bigvee_a g(b,a)\&f(a,b') = \bigvee_a Eb\&f(a,b)\&f(a,b') \leq Eb\&[\![b=b']\!] = [\![b=b']\!].$$

Hence, the reverse composite yields the identity map on B. The map of quantal sets

$$f : A \to B$$

is therefore an isomorphism, of which the inverse has been explicitly constructed by the assignment defined above. ▯

Finally, we note the following:

Corollary. *For any right Gelfand quantale Q, the quantal set*

$$\mathbb{1}_Q$$

is the terminal object in the category of quantal sets over Q.

For, it has already been remarked that, for any quantal set A, there exists a map

$$\tau_A : A \to \mathbb{1}_Q$$

in the category of quantal sets over Q. That this is the unique such map for any quantal set A may be seen by noting that any map

$$t : A \to \mathbb{1}_Q$$

has the property that $t(a,q) = t(a,q)\&Eq = t(a,q)\&q$ for each $a \in A$ and each $q \in Q$. Then, $t(a,q) = t(a,q)\&q\&1 \leq t(a,1)$ by extensionality in $\mathbb{1}_Q$; hence, $t(a,q) = t(a,q)\&q \leq t(a,1)\&q$. But, $t(a,1)\&q = t(a,1)\&1\&q \leq t(a,q)$ by extensionality again; hence, $t(a,q) = t(a,1)\&q$ for each $a \in A$ and each $q \in Q$. Now, $\bigvee_q t(a,q) = Ea$, by the totality of the map; hence $t(a,1) = Ea$, because one has that $t(a,q) \leq t(a,1)$ for all $q \in Q$. Thus,

$$t(a,q) = Ea\&q$$

for each $a \in A$ and each $q \in Q$. So the map considered is exactly the map

$$\tau_A : A \to \mathbb{1}_Q,$$

which is therefore unique. ▯

3. Complete Q-Sets

Once a concept of quantal set is given, one approach to the idea of a sheaf on the quantale, extending that developed in the context of locales [6], lies in determining an appropriate notion of completeness for a quantal set. Explicitly, one wishes to describe firstly the notion of a subset of a quantal set, of which certain subsets will then be defined to be the singletons. A quantal set will then be considered complete provided that each singleton subset is determined by a unique element of the set. Towards this end we make first the following:

Definition. By a *subset* of a quantal set A over a right Gelfand quantale Q will be meant a mapping

$$s : A \to Q$$

which satisfies the following conditions:

i) $s(a)\&[\![a = a']\!] \leq s(a')$;

ii) $s(a) \leq s(a)\&Ea$,

for each $a, a' \in A$.

Example. Consider the quantal set

$$\mathbb{1}_Q$$

which is the terminal object in the category of quantal sets over Q. For each $p \in Q$, consider the assignment

$$\tilde{p} : Q \to Q$$

defined by:

$$\tilde{p}(q) = p\&q$$

for each $q \in Q$. Then this yields a subset of $\mathbb{1}_Q$. For, the first condition is satisfied since:

$$p\&q\&q\&q' \leq p\&q',$$

by the right identity of the quantale, while the second:

$$p\&q \leq p\&q\&q,$$

is satisfied by the idempotency of the quantale. So, each element of the quantale Q determines a subset of the quantal set $\mathbb{1}_Q$. In fact, it will be shown later that each subset of $\mathbb{1}_Q$ is obtained in this way for a unique $p \in Q$. The subsets of the quantal set $\mathbb{1}_Q$ therefore correspond precisely to the elements of the quantale Q.

The subsets of $\mathbb{1}_Q$ just described are particular instances of the following concept:

Definition. By a *singleton* of a quantal set A over a quantale Q will be meant a subset

$$s : A \to Q$$

of A which satisfies the condition that:

$$Ea\&s(a)\&s(a') \leq [\![a = a']\!],$$

for each $a, a' \in A$.

Example. For any quantal set A over Q, each element $a \in A$ determines a subset

$$\tilde{a} : A \to Q$$

by the assignment:

$$\tilde{a}(b) = [\![a = b]\!]$$

for each $b \in A$. The subset of A obtained is a singleton of the quantal set A. It is a subset, because $[\![a = b]\!]\&[\![b = b']\!] \leq [\![a = b']\!]$ and $[\![a = b]\!] \leq [\![a = b]\!]\&Eb$ for all $b, b' \in A$. Moreover, it is a singleton, because

$$Eb\&[\![a = b]\!]\&[\![a = b']\!] \leq [\![b = a]\!]\&[\![a = b']\!] \leq [\![b = b']\!],$$

for all $b, b' \in A$. It may be remarked that the right symmetricity,

$$Eb\&[\![a = b]\!] \leq [\![b = a]\!]$$

of equality in a quantal set is exactly what is required in showing that this subset is indeed a singleton of A.

It may be noted generally that any subset

$$s : A \to Q$$

of a quantal set A necessarily satisfies the condition:

$$s(a)\&Ea = s(a)$$

for each $a \in A$. In the case of a singleton subset of a quantal set A, one may see moreover that:

$$s(a)\&s(b) = s(b)\&s(a)$$

for each $a, b \in A$. For,

$$\begin{aligned} s(a)\&s(b) &= s(a)\&s(b)\&Eb \\ &= s(a)\&Eb\&s(b) \\ &= s(a)\&Eb\&s(b)\&s(a)\&s(a) \\ &\leq s(a)\&[\![b = a]\!]\&s(a) \\ &= s(a)\&[\![a = b]\!]\&s(a) \\ &\leq s(b)\&s(a), \end{aligned}$$

from which the conclusion follows on interchanging $a, b \in A$.

With these preliminaries we may make the following:

Definition. A quantal set A over a right Gelfand quantale Q will be said to be *complete* provided that each singleton of A is of the form

$$\tilde{a} : A \to Q$$

for some unique $a \in A$.

EXAMPLE. For any quantale Q, the quantal set $\mathbb{1}_Q$ is complete. Indeed, it will be shown that any subset of $\mathbb{1}_Q$ is determined by a unique element of Q. For, given any subset

$$s : Q \to Q$$

of the quantal set $\mathbb{1}_Q$, consider the element

$$p = \bigvee_q \ s(q)$$

of the quantale Q. It is asserted that

$$s(q) = p\&q$$

for each $q \in Q$. For certainly one has that $s(q) \le p$, by the construction of $p \in Q$; hence that $s(q) = s(q)\&q \le p\&q$. But, equally,

$$p\&q = \bigvee_{q'} s(q')\&q = \bigvee_{q'} s(q')\&q'\&q = \bigvee_{q'} s(q')\&[\![q' = q]\!] \le s(a).$$

Hence, $s(q) = p\&q = [\![p = q]\!] = \tilde{p}(q)$ for each $q \in Q$. The uniqueness of the element $p \in Q$ having this property is given by the observation that

$$s(1) = [\![p = 1]\!] = p\&1 = p.$$

The quantal set $\mathbb{1}_Q$ is therefore complete.

Now, for any quantal set A over the quantale Q, consider the set

$$S(A)$$

of all singletons of A. Define the extent and the equality of singletons of A in the following manner:

$$Es = \bigvee_{a \in A} \ s(a),$$

and $$[\![s = t]\!] = \bigvee_{a \in A} \ s(a)\&t(a),$$

for each $s, t \in S(A)$. Then $S(A)$ is a quantal set. For, given $s \in S(A)$ one has that:

$$[\![s = t]\!] \ = \bigvee_a s(a)\&t(a) \le \bigvee_{a,a'} s(a)\&t(a') = \bigvee_a s(a) \& \bigvee_{a'} t(a') = Es\&Et \ .$$

And, for any $s, t \in S(A)$, one has that:

$$[\![s = t]\!] = \bigvee_a s(a)\&t(a) \le \bigvee_{a,a'} s(a)\&t(a') = Es\&Et.$$

Moreover, given $s, t \in S(A)$, one has that:

$$\begin{aligned} Et\&[\![s = t]\!] &= \bigvee_{a,a'} \ t(a)\&s(a')\&t(a') \\ &= \bigvee_{a,a'} \ t(a)\&t(a)\&s(a')\&t(a') \\ &= \bigvee_{a,a'} \ t(a)\&s(a')\&t(a')\&t(a) \\ &= \bigvee_{a,a'} \ t(a)\&s(a')\&Ea'\&t(a')\&t(a) \\ &\le \bigvee_{a,a'} \ t(a)\&s(a')\&[\![a' = a]\!] \end{aligned}$$

$$\begin{aligned} &\le \bigvee_a \ t(a)\&s(a) \\ &= [\![t = s]\!]. \end{aligned}$$

Finally, given $s, t, u \in S(A)$, one argues similarly that:

$$\begin{aligned} [\![s = t]\!]\&[\![t = u]\!] &= \bigvee_{a,a'} \ s(a)\&t(a)\&t(a')\&u(a') \\ &= \bigvee_{a,a'} \ s(a)\&Ea\&t(a)\&t(a')\&u(a') \\ &\le \bigvee_{a,a'} \ s(a)\&[\![a = a']\!]\&u(a') \\ &\le \bigvee_{a'} \ s(a')\&u(a') \\ &= [\![s = u]\!], \end{aligned}$$

which shows that $S(A)$ is a quantal set.

REMARK. For any element $b \in A$ of the quantal set A, the singleton $\tilde{b} \in S(A)$ which it determines has equality with any other singleton $s \in S(A)$ which satisfies:

$$[\![s = \tilde{b}]\!] = s(b).$$

For, in one direction, one has that:

$$[\![s = \tilde{b}]\!] = \bigvee_a \ s(a)\&[\![b = a]\!] = \bigvee_a \ s(a)\&[\![a = b]\!] \le s(b),$$

while, conversely, one sees that:

$$s(b) = s(b)\&Eb = s(b)\&[\![b = b]\!] \le \bigvee_a \ s(a)\&[\![b = a]\!] = [\![s = \tilde{b}]\!].$$

In particular, it may be remarked that:

$$[\![\tilde{a} = \tilde{b}]\!] = [\![a = b]\!]$$

for any $a, b \in A$. Hence,

$$E\tilde{a} = Ea$$

for any $a \in A$.

With these preliminaries, we may now state the following:

THEOREM. *For any right Gelfand quantale Q, there are adjoint functors*

$$Q\textbf{-Sets} \ \rightleftarrows \ \textbf{Complete } Q\textbf{-Sets},$$

of which the adjoint to the inclusion of the category of complete quantal sets in that of quantal sets assigns to each quantal set A the quantal set S(A) of singletons of the quantal set A.

To prove the assertion of the theorem, it will be shown firstly that the quantal set $S(A)$ is indeed complete. Moreover, that there exists a map of quantal sets over Q :

$$\eta_A : A \to S(A)$$

which is universal amongst maps from A to complete quantal sets. From this, the assertion of the theorem follows.

So, consider firstly a singleton

$$\sigma : S(A) \to Q$$

of the quantal set $S(A)$. It is asserted that the mapping

$$s : A \to Q,$$

defined by

$$s(a) = \sigma(\tilde{a})$$

for each $a \in A$, yields uniquely an element $s \in S(A)$ for which $\tilde{s} = \sigma$.

That s is a singleton of the quantal set A follows trivially from the conditions satisfied because σ is a singleton of the quantal set $S(A)$, by recalling that $[\![a = a']\!] = [\![\tilde{a} = \tilde{a}']\!]$ and $Ea = E\tilde{a}$ for each $a, a' \in A$. But then, one sees that

$$\begin{aligned} \sigma(t) &= \sigma(t)\&Et \\ &= \sigma(t)\& \bigvee_a t(a) \\ &= \bigvee_a \sigma(t)\&[\![t = \tilde{a}]\!]\&t(a) \\ &\le \bigvee_a \sigma(\tilde{a})\&t(a) \\ &= \bigvee_a s(a)\&t(a) \\ &= [\![s = t]\!], \end{aligned}$$

while, conversely,

$$\begin{aligned} [\![s = t]\!] &= \bigvee_a s(a)\&t(a) \\ &= \bigvee_a \sigma(\tilde{a})\&t(a) \\ &= \bigvee_a \sigma(\tilde{a})\&[\![t = \tilde{a}]\!] \\ &= \bigvee_a \sigma(\tilde{a})\&[\![\tilde{a} = t]\!] \\ &\le \sigma(t), \end{aligned}$$

from which the rest of the required result follows. The uniqueness of the element $s \in S(A)$ is immediate, because

$$s(a) = [\![s = \tilde{a}]\!] = \sigma(\tilde{a}) = [\![s' = \tilde{a}]\!] = s'(a)$$

for each $a \in A$, for any singleton $s' \in S(A)$ satisfying the required condition. Hence, the quantale set $S(A)$ is complete.

To obtain a map of quantal sets

$$\eta_A : A \to S(A),$$

consider the mapping from $A \times S(A)$ to the quantale Q, given by defining

$$\eta_A(a, s) = [\![\tilde{a} = s]\!]$$

for each $a \in A$ and $s \in S(A)$. It may be remarked that one has, firstly, that:

$$\eta_A(a, s) = [\![\tilde{a} = s]\!] \le E\tilde{a}\&Es = Ea\&Es.$$

Then, that:

$$[\![a = a']\!]\&\eta_A(a', s) = [\![\tilde{a} = \tilde{a}']\!]\&[\![\tilde{a}' = s]\!] \le [\![\tilde{a} = s]\!] = \eta_A(a, s),$$

and $$\eta_A(a, s)\&[\![s = s']\!] = [\![\tilde{a} = s]\!]\&[\![s = s']\!] \le [\![\tilde{a} = s']\!] = \eta_A(a, s').$$

Finally, that

$$\begin{aligned} Es\&Es'\&\eta_A(a, s)\&\eta_A(a, s') &= Es\&Es'\&[\![\tilde{a} = s]\!]\&[\![\tilde{a} = s']\!] \\ &= Es\&[\![s = \tilde{a}]\!]\&[\![\tilde{a} = s']\!]\&Es' \\ &= [\![s = \tilde{a}]\!]\&[\![\tilde{a} = s']\!] \\ &\le [\![s = s']\!] \end{aligned}$$

and $$Ea = E\tilde{a} = [\![\tilde{a} = \tilde{a}]\!] \le \vee_s [\![\tilde{a} = s]\!] = \bigvee_s \eta_A(a, s).$$

Hence, one has indeed a map of quantal sets from A to $S(A)$.

Now, suppose given a map

$$\varphi : A \to B$$

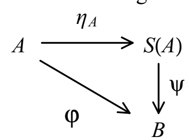
from the quantal set A to a complete quantal set B. Consider the mapping

$$\psi : S(A) \times B \to Q$$

defined by setting $\psi(s, b) = \bigvee_a s(a)\&\varphi(a, b)$ for each $s \in S(A)$ and $b \in B$. It may be proved straightforwardly that this determines a map of quantal sets

$$\psi : S(A) \to B\,.$$

Moreover, this makes the diagram:



commute, and is unique in this respect.

To see that the diagram is commutative, observe firstly that:

$$[\![a = a']\!] = \bigvee_s [\![\tilde{a} = s]\!]\&[\![s = \tilde{a}']\!]$$

for each $a, a' \in A$, by the transitivity of equality on $S(A)$ and the observation that

$$[\![\tilde{a} = \tilde{a}']\!] = [\![a = a']\!].$$

Then

$$\begin{aligned} \bigvee_s \eta_A(a,s) \,\&\, \psi(s,b) &= \bigvee_s [\![\tilde{a} = s]\!] \,\&\, s(a) \,\&\, \varphi(a,b) \\ &= \bigvee_s [\![\tilde{a} = s]\!] \,\&\, [\![s = \tilde{a}]\!] \,\&\, \varphi(a,b) \\ &= Ea \,\&\, \varphi(a,b) \\ &= \varphi(a,b) \end{aligned}$$

for each $a \in A$ and $b \in B$. While, to see the uniqueness, note that:

$$[\![s = t]\!] = \bigvee_a [\![s = \tilde{a}]\!] \,\&\, [\![\tilde{a} = t]\!].$$

Then, for any map $\psi : S(A) \to B$ for which $\varphi(a,b) = \vee_t \, \eta_A(a,t) \,\&\, \psi(t,b)$ for each $a \in A$ and $b \in B$, one has that:

$$\begin{aligned} s(a) \,\&\, \varphi(a,b) &= \bigvee_a \bigvee_t \, s(a) \,\&\, \eta_A(a,t) \,\&\, \psi(t,b) \\ &= \bigvee_t \bigvee_a [\![s = \tilde{a}]\!] \,\&\, [\![\tilde{a} = t]\!] \,\&\, \psi(t,b) \\ &= \bigvee_t [\![s = t]\!] \,\&\, \psi(t,b) \\ &= \psi(s,b), \end{aligned}$$

for each $s \in S(A)$ and $b \in B$.

The map considered is therefore unique, from which the existence of the adjointness and the naturality of the maps concerned then follows. ▯

The functor that takes any quantal set A to its completion $S(A)$ will be denoted by

$$S : Q\textbf{-Sets} \to \textbf{Complete } Q\textbf{-Sets},$$

whilst the embedding of the full subcategory of complete quantal sets in the category of quantal sets will be denoted by

$$I : \textbf{Complete } Q\textbf{-Sets} \to Q\textbf{-Sets}.$$

In particular, the adjunction and coadjunction maps will henceforth be considered to be written as

$$\eta_A : A \to IS(A)$$

and

$$\varepsilon_B : SI(B) \to B$$

for any quantal set A and any complete quantal set B.

It may be remarked immediately that we have proved more than simply the existence of adjoint functors. It will be recalled that the adjunction

$$\eta_A : A \to S(A)$$

was defined by the assignment

$$\eta_A(a,s) = [\![\tilde{a} = s]\!]$$

for each $a \in A$ and $s \in S(A)$. In the course of the proof, it has been noted that:

$$\bigvee_s [\![\tilde{a} = s]\!] \,\&\, [\![s = \tilde{a}']\!] = [\![a = a']\!]$$

for each $a, a' \in A$; and that

$$\bigvee_a [\![s = \tilde{a}]\!] \,\&\, [\![\tilde{a} = t]\!] = [\![s = t]\!]$$

for each $s, t \in S(A)$. These observe that the map of quantal sets

$$\varepsilon_A : S(A) \to A$$

defined by the assignment

$$\varepsilon_A(s,a) = [\![s = \tilde{a}]\!],$$

for each $s \in S(A)$ and $a \in A$, is indeed an inverse to the adjunction. Hence, one has the following:

COROLLARY. *For any right Gelfand quantale Q, the adjoint functors*

$$Q\textbf{-Sets} \underset{I}{\overset{S}{\rightleftarrows}} \textbf{Complete } Q\textbf{-Sets}$$

establish an equivalence of categories. ▯

Finally, it may be remarked that the definition of the map

$$S(f) : S(A) \to S(B)$$

of complete quantal sets determined by any map $f : A \to B$ of quantal sets may be reformulated slightly in the light of these results. The map was defined to be that given by the relation

$$S(f) : S(A) \times S(B) \to Q$$

that assigns to each pair $(s,t) \in S(A) \times S(B)$ of singletons the element

$$S(f)(s,t) = \bigvee_{a,b} \, s(a) \,\&\, f(a,b) \,\&\, \eta_B(b,t)$$

of the quantale Q. By the remarks following the definition of the concept of a map of quantal sets, one has that $f(a,b) = f(a,b) \,\&\, Eb$. Moreover, $Eb \,\&\, \eta_B(b,t) = Eb \,\&\, t(b)$, since, on the one hand,

$$\begin{aligned} Eb \,\&\, \eta_B(b,t) &= Eb \,\&\, [\![\tilde{b} = t]\!] \\ &= \bigvee_c \, Ec \,\&\, [\![b = c]\!] \,\&\, t(c) \\ &= \bigvee_c \, Ec \,\&\, t(c) \,\&\, [\![b = c]\!] \end{aligned}$$

$$\leq Eb \,\&\, t(b),$$

while, on the other,

$$\begin{aligned} Eb \,\&\, t(b) &= Eb \,\&\, Eb \,\&\, Eb \,\&\, t(b) \\ &\leq Eb \,\&\, [\![b = b]\!] \,\&\, t(b) \\ &\leq \bigvee_c Eb \,\&\, [\![b = c]\!] \,\&\, t(c) \\ &= Eb \,\&\, \eta_B(b,t). \end{aligned}$$

Hence, $\quad S(f)(s,t) = \bigvee_{a,b} s(a) \,\&\, f(a,b) \,\&\, t(b)$, as asserted.

In conclusion, it may be remarked that this equivalence exactly extends to the context of a quantale the equivalence which exists in the case of a locale. The concept of completeness of a quantal set evidently reduces in the case of a locale to that already known, giving the equivalence between local sets and complete local sets which is known, if a little unexpected when first encountered, in that situation. Hence, one has in particular that:

Corollary. *For any locale Q, the adjoint functors*

$$Q\text{-}\mathbf{Sets} \underset{I}{\overset{S}{\rightleftarrows}} \mathbf{Complete}\ Q\text{-}\mathbf{Sets}$$

are exactly the canonical adjoint equivalence between the category of local sets and the category of complete local sets over the locale Q. ▯

4. Presheaves on Q

The concept of completeness for a quantal set was introduced to motivate the definition of the notion of a sheaf on a quantale. In the context of locales, these concepts are known to coincide in a very strict sense: any complete local set over a locale is canonically a sheaf, and every sheaf is canonically a complete local set over the locale. The expectation, therefore, is that this coincidence can be maintained on extending these ideas to the case of right Gelfand quantales. The first step towards establishing such a correspondence is to use the notion of complete quantal set already established to motivate the concept of presheaf on a quantale Q, before considering the condition for a presheaf on the quantale to be a sheaf.

Recalling that any complete quantal set A is isomorphic to its completion $S(A)$ by the map of quantal sets induced by the mapping

$$A \to S(A)$$

that assigns to each $a \in A$ the singleton $\tilde{a} \in S(A)$ that it represents uniquely, we begin by examining the case of a quantal set of that form:

Example. For any quantal set A, given any singleton $s \in S(A)$ on A, and any $p \in Q$, the mappings

$$p \downharpoonleft s : A \to Q \text{ and } s \upharpoonleft p : A \to Q$$

defined by

$$(p \downharpoonleft s)(a) = p \,\&\, s(a) \text{ and } (s \upharpoonleft p)(a) = s(a) \,\&\, p$$

for each $a \in A$, respectively, again define singleton subsets of A. Moreover, these restrictions of a singleton $s \in S(A)$ by any $p \in Q$ satisfy the conditions that:

$$\begin{aligned} &p \downharpoonleft q \downharpoonleft s = p\&q \downharpoonleft s && s \upharpoonleft p \upharpoonleft q = s \upharpoonleft p\&q \\ &Es \downharpoonleft s = s && s \upharpoonleft Es = s \\ &E(p \downharpoonleft s) = p\&Es && E(s \upharpoonleft p) = Es\&p \end{aligned}$$

and $\qquad (p \downharpoonleft s) \upharpoonleft q = p \downharpoonleft (s \upharpoonleft q),$

for any $s \in S(A)$ and any $p, q \in Q$.

That the mappings $p \downharpoonleft s$ and $s \upharpoonleft p$ define subsets of the quantal set A may be verified straightforwardly from the definitions by applying the corresponding properties of the subset $s \in S(A)$, together with, in the case of $s \upharpoonleft p$, the right symmetricity of the quantale Q. Thus, for instance:

$$(s \upharpoonleft p)(a) \& [\![a = a']\!] = s(a) \& p \& [\![a = a']\!] = s(a) \,\&\, [\![a = a']\!] \& p \leq s(a') \,\&\, p = (s \upharpoonleft p)(a')\ .$$

That the subsets are again singletons follows similarly by invoking the corresponding property of the subset $s \in S(A)$ and applying the rightsidedness of the elements of the quantale Q. Thus, for instance:

$$Ea \& (s \upharpoonleft p)(a) \& (s \upharpoonleft p)(a') = Ea \& s(a) \& p \& s(a') \& p \leq Ea \& s(a) \& s(a') \leq [\![a = a']\!]\ .$$

Hence, both the left and the right restrictions of any singleton subset $s \in S(A)$ are again singletons on the quantal set A.

The main point that emerges from considering this example is that the completion $S(A)$ of a quantal set A admits both a left restriction and a right restriction for any element $p \in Q$, with the expectation that these restrictions will be distinct unless the quantale Q is actually commutative. With this in mind, we arrive at the following:

Definition. By a *presheaf A* on a right Gelfand quantale Q will be meant a set A together with mappings

$$\downharpoonleft : Q \times A \to A \text{ and } \upharpoonleft : A \times Q \to A,$$

which will be referred to as respectively as *left restriction* and *right restriction* over Q, and a mapping

$$E : A \to Q$$

which will be referred to as the *extent* of A, together satisfying the conditions that:

$$p \downharpoonleft q \downharpoonleft a = p\&q \downharpoonleft a \qquad a \upharpoonleft p \upharpoonleft q = a \upharpoonleft p\&q$$

$$Ea \downharpoonleft a = a \qquad a \upharpoonleft Ea = a$$

$$E(p \downharpoonleft a) = p\&Ea \qquad E(a \upharpoonleft p) = Ea\&p$$

and $$(p \downharpoonleft a) \upharpoonleft q = p \downharpoonleft (a \upharpoonleft q),$$

for any $a \in A$ and any $p, q \in Q$.

Now consider any complete quantal set A over the quantale Q, and observe that we have the following:

Construction. For each element $a \in A$ of the complete quantal set A, consider the singleton subset $\tilde{a} \in S(A)$ defined by

$$\tilde{a}(b) = [\![a = b]\!]$$

for each $b \in A$. Note that, by the completeness of A, each singleton subset of A is of this form for a unique element of A. For any element $p \in Q$, denote by

$$p \downharpoonleft a \text{ and } a \upharpoonleft p$$

respectively the unique elements of A that represent the singleton subsets

$$p \downharpoonleft \tilde{a} \text{ and } \tilde{a} \upharpoonleft p$$

defined above. It is asserted that, with respect to these operations of left and right restriction, together with the extent operator of the quantal set A, the underlying set of any complete quantal set A is indeed a presheaf on the quantale Q, which we shall denote by $\Gamma(A)$.

For, observe firstly that, by the definition of the left and the right restrictions of a complete quantal set, for any element $a \in A$ and for any $p \in Q$ the element $p \downharpoonleft a \in A$ is uniquely determined by the property that:

$$[\![p \downharpoonleft a = b]\!] = p\&[\![a = b]\!]$$

for all $b \in A$. Similarly, the element $a \upharpoonleft p \in A$ uniquely satisfies the condition that:

$$[\![a \upharpoonleft p = b]\!] = [\![a = b]\!]\&p$$

for all $b \in A$. From this observation, the required conditions on the left restriction may be proved by noting respectively that $[\![q \downharpoonleft p \downharpoonleft a = b]\!] = q\&[\![p \downharpoonleft a = b]\!] = q\&p\&[\![a = b]\!] = [\![q\&p \downharpoonleft a = b]\!]$, and that $E(p \downharpoonleft a) = [\![p \downharpoonleft a = p \downharpoonleft a]\!] = p\&[\![a = p \downharpoonleft a]\!] = p\&[\![p \downharpoonleft a = a]\!] = p\&p\&[\![a = a]\!] = p\&Ea$ for all $b \in A$, by the right symmetricity of equality in a quantal set, with similar remarks in the case of the right restriction.

With regard to the condition concerning the extent, it may be noted that in the case of left restriction, one has that $[\![Ea \downharpoonleft a = b]\!] = Ea\&[\![a = b]\!] = [\![a = b]\!]$ for each $b \in A$, giving the required equality, with a similar argument on the right. Finally, the condition that implies the associativity of left and right restrictions follows from that of the product of the quantale by

noting that $[\![p \downharpoonleft (a \upharpoonleft q) = b]\!] = p\&[\![a \upharpoonleft q = b]\!] = p\&[\![a = b]\!]\&q = [\![p \downharpoonleft a = b]\!]\&q = [\![(p \downharpoonleft a) \upharpoonleft q = b]\!]$ for all $b \in A$, establishing that the complete quantal set A is indeed canonically a presheaf on Q.

Recalling the characterisation of the restrictions given above, we have the following:

Definition. By the *canonical presheaf*

$$\Gamma(A)$$

determined by a complete quantal set A over the right Gelfand quantale Q will be meant the presheaf of which the underlying set and the extent operator are those of the quantal set A, and of which the left and right restrictions are uniquely determined by requiring that

$$[\![p \downharpoonleft a = b]\!] = p\&[\![a = b]\!]$$

and $$[\![a \upharpoonleft p = b]\!] = [\![a = b]\!]\&p$$

for any $a, b \in A$ and $p \in Q$.

It may be remarked in passing that, although we had already shown that the completion $S(A)$ of any quantal set A was canonically a presheaf, the existence of the canonical isomorphism

$$\eta_A : A \to S(A)$$

is not sufficient to imply that A is again a presheaf. Just as completeness for a quantal set A requires the existence of certain elements in its underlying set A , so the existence of the restrictions of an element requires the presence of the elements concerned. Indeed, the completeness of the quantal set A is exactly what is needed in order to ensure that the restrictions existing in the completion are represented within the quantal set A.

It may also be remarked that although the description of these restriction mappings is straightforward to define, it may still be difficult in any particular case to obtain an intrinsic characterisation of the restrictions of an element, since this involves passing through the quantal set $S(A)$ of singletons on the quantal set A.

There is, however, one immediate case in which this can be computed directly and naturally, yielding in particular an instance of a presheaf on which the left and right restrictions are distinct unless the quantale Q is commutative, namely the following:

Example. The quantal set 1_Q, which is the terminal object in the category of quantal sets over Q, obtained by taking the set Q, together with the equality and extent given by:

$$[\![p = q]\!] = p\&q \text{ and } Ep = p$$

for each $p, q \in Q$, is a presheaf with respect to the restriction mappings given by

$$p \downharpoonleft q = p\&q \text{ and } q \upharpoonleft p = q\&p$$

for each $p, q \in Q$.

To see this, it is enough to check that the element $p\&q \in Q$ indeed represents each of the singleton subsets $p \downharpoonleft \tilde{q}$ and $\tilde{p} \upharpoonleft q$ of the quantal set $\mathbb{1}_Q$. However, in establishing the completeness of the quantal set $\mathbb{1}_Q$, we have shown that each singleton subset

$$s : Q \to Q$$

of this particular quantal set is represented uniquely by the element $s(1) \in Q$ which is the image of the top element $1 \in Q$. Hence, from the definition of the singleton subset $\tilde{q} \in S(Q)$ determined by an element $q \in Q$, and by the rightsidedness of the quantale Q, we have that

$$p \downharpoonleft q = (p \downharpoonleft \tilde{q})(1) = p\&\tilde{q}(1) = p\&[\![q = 1]\!] = p\&q\&1 = p\&q,$$

and that $$q \upharpoonleft p = (\tilde{q} \upharpoonleft p)(1) = \tilde{q}(1)\&p = [\![q = 1]\!]\&p = q\&1\&p = q\&p$$

for any $p \in Q$, yielding the required assertion.

Recalling that, by the equivalence between the category of quantal sets over a quantale Q and that of complete quantal sets over Q, any map

$$f : A \to B$$

of quantal sets over Q is equivalent to the map

$$S(f) : S(A) \to S(B)$$

of completions that it determines, in the sense that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & IS(A) \\ f \downarrow & & \downarrow IS(f) \\ B & \xrightarrow[\eta_B]{} & IS(B) \end{array} ,$$

in which the maps η_A and η_B are the canonical isomorphisms from a quantal set to its completion, is commutative, we may begin to consider the concept of a map of presheaves on Q by examining the case of a map of complete quantal sets determined in this way.

Observing firstly that, by the remarks made in the preceding section concerning the completion functor, the map

$$S(f) : S(A) \to S(B)$$

considered is that given by the relation

$$S(f) : S(A) \times S(B) \to Q$$

defined by

$$S(f)(s,t) = \bigvee_{a,b} s(a)\&f(a,b)\&t(b)$$

for any $s \in S(A)$ and $t \in S(B)$, we consider the following:

Example. For any map

$$f : A \to B$$

of quantal sets over Q, the assignment to each singleton $s \in S(A)$ of the mapping

$$\varphi(s) : B \to Q$$

defined at each $b \in B$ by taking the element

$$\varphi(s)(b) = \bigvee_a s(a)\&f(a,b)$$

of the quantale Q yields a mapping

$$\varphi : S(A) \to S(B)$$

from the set $S(A)$ of singletons on A to the set $S(B)$ of singletons on B. Moreover, the mapping thereby obtained satisfies the conditions that:

$$p \downharpoonleft \varphi(s) = \varphi(p \downharpoonleft s) \qquad \varphi(s) \upharpoonleft p = \varphi(s \upharpoonleft p)$$

and $$E\varphi(s) = Es$$

for each singleton $s \in S(A)$ and any $p \in Q$.

That the mapping

$$\varphi(s) : B \to Q$$

defines a singleton subset $\varphi(s) \in S(B)$ may be verified in the following manner: it is a subset of B, since

$$\begin{aligned} \varphi(s)(b)\&[\![b = b']\!] &= \bigvee_a s(a)\&f(a,b)\&[\![b = b']\!] \\ &\leq \bigvee_a s(a)\&f(a,b') \\ &= \varphi(s)(b'), \end{aligned}$$

and $$\varphi(s)(b) = \bigvee_a s(a)\&f(a,b) \leq \bigvee_a s(a)\&f(a,b)\&Eb = \varphi(s)(b)\&Eb;$$

and it is a singleton of A, since

$$\begin{aligned} Eb\&\varphi(s)(b)\&\varphi(s)(b') &= \bigvee_{a,a'} Eb\&s(a)\&f(a,b)\&s(a')\&f(a',b') \\ &= \bigvee_{a,a'} Eb\&f(a,b)\&Ea\&s(a)\&s(a')\&f(a',b') \\ &\leq \bigvee_{a,a'} Eb\&f(a,b)\&[\![a = a']\!]\&f(a',b') \\ &\leq \bigvee_a Eb\&f(a,b)\&f(a,b') \\ &\leq [\![b = b']\!]. \end{aligned}$$

Thus, assigning to each element $s \in S(A)$ the element $\varphi(s) \in S(B)$ indeed yields a mapping

$$\varphi : S(A) \to S(B)$$

from the set of singletons on A to the set of singletons on B.

That this mapping satisfies the conditions asserted with respect to restriction and extent may be verified by observing that

$$(p \downharpoonleft \varphi(s))(b) = p\&\varphi(s)(b) = p\& \bigvee_a s(a)\&f(a,b) = \bigvee_a p\& s(a)\&f(a,b)$$
$$= \bigvee_a (p \downharpoonleft s)(a)\&f(a,b) = \varphi(p \downharpoonleft s)(b)$$

for each $b \in B$, giving the condition for left restriction, with a similar argument in the case of right restriction, and that

$$E\varphi(s) = \bigvee_b \varphi(s)(b) = \bigvee_{a,b} s(a)\&f(a,b) = \bigvee_a s(a)\&(\bigvee_b f(a,b))$$
$$= \bigvee_a s(a)\& Ea = \bigvee_a s(a) = Es$$

in the case of extent.

With these observations in mind, we make the following:

Definition. By a *map of presheaves*

$$\varphi : A \to B$$

on a right Gelfand quantale Q will be meant a mapping from the set A to the set B satisfying the following conditions with respect to the respective restriction and extent mappings:

$$\varphi(p \downharpoonleft a) = p \downharpoonleft \varphi(a) \qquad \varphi(a \upharpoonleft p) = \varphi(a) \upharpoonleft p$$

and

$$E\varphi(a) = Ea$$

for each $a \in A$ and for each $p \in Q$.

Now consider any map $f : A \to B$ of complete quantal sets over the quantale Q, and observe that we have the following:

Construction. For each element $a \in A$ consider the singleton subset $\tilde{a} \in S(A)$ of A, which is therefore mapped by the map of presheaves

$$\varphi : S(A) \to S(B)$$

defined above to a singleton subset $\varphi(\tilde{a}) \in S(B)$ of B. Denoting by

$$f(a) \in B$$

the unique element of the complete quantal set B that represents this singleton subset of B, it is asserted that the mapping

$$f : A \to B$$

from the underlying set of A to the underlying set of B thereby determined yields a map of presheaves from the canonical presheaf $\Gamma(A)$ to the canonical presheaf $\Gamma(B)$, which we shall denote by $\Gamma(f)$.

For, observe firstly that the element $f(a) \in B$ described above may equally be characterised by the condition that

$$[\![f(a) = b]\!] = f(a,b)$$

for each $b \in B$. For, on the one hand,

$$[\![f(a) = b]\!] = \varphi(\tilde{a})(b) = \bigvee_{a'} [\![a = a']\!]\&f(a',b) \le f(a,b),$$

while, on the other, we have that

$$f(a,b) = Ea\&f(a,b) = [\![a = a]\!]\&f(a,b)$$
$$\le \bigvee_{a'} [\![a = a']\!]\&f(a',b) = \varphi(\tilde{a})(b) = [\![f(a) = b]\!],$$

which together yield the equality asserted. Extending to this context the convention that a mapping and the functional relation that defines are denoted by the same symbol, we write:

$$f : A \to B$$

for the mapping from the set A to the set B thereby obtained.

It is asserted that this mapping indeed determines a map

$$\Gamma(f) : \Gamma(A) \to \Gamma(B)$$

of presheaves on Q. For, recalling that for any $a \in A$ and any $p \in Q$, the element $p \downharpoonleft a$ is uniquely determined by the property that

$$[\![p \downharpoonleft a = a']\!] = p\&[\![a = a']\!]$$

for each $a' \in A$, it follows that $f(p \downharpoonleft a) = p \downharpoonleft f(a)$ for any $a \in A$ and $p \in Q$. For, by the above remarks, it suffices to show that $f(p \downharpoonleft a, b) = p\&f(a,b)$ for any $b \in B$. But, on the one hand,

$$f(p \downharpoonleft a) = E(p \downharpoonleft a)\&f(p \downharpoonleft a,b) = p\&Ea\&f(p \downharpoonleft a,b)$$
$$= p\&p\&[\![a = a]\!]\&f(p \downharpoonleft a,b)$$
$$= p\&[\![p \downharpoonleft a = a]\!]\&f(p \downharpoonleft a,b)$$
$$= p\&[\![a = p \downharpoonleft a]\!]\&f(p \downharpoonleft a,b)$$
$$\le p\&f(a,b);$$

while conversely:

$$p\&f(a,b) = p\&Ea\&f(a,b) = p\&[\![a = a]\!]\&f(a,b)$$
$$= [\![p \downharpoonleft a = a]\!]\&f(a,b) \le f(p \downharpoonleft a,b),$$

giving the required equality. The condition for right restriction may be established similarly, while, in the case of the extent, we observe that for any $b \in B$ we have that:

$$f(a,b) = f(a,b)\&f(a,b) = f(a,b)\&[\![f(a) = b]\!] = f(a,b)\&[\![b = f(a)]\!] \le f(a,f(a)),$$

since $f(a) \in B$ evidently implies that $f(a,f(a)) \le \vee_b f(a,b)$, establishing that this determines a canonical map $\Gamma(f) : \Gamma(A) \to \Gamma(B)$ of presheaves on Q.

Recalling the characterisation of the underlying mapping of the map of presheaves obtained, we have the following:

Definition. By the *canonical map of presheaves*

$$\Gamma(f) : \Gamma(A) \to \Gamma(B)$$

determined by a map $f : A \to B$ of complete quantal sets over the right Gelfand quantale Q will be meant the map of which the underlying mapping is uniquely determined by requiring that

$$[\![f(a) = b]\!] = f(a, b)$$

for each $a \in A$ and $b \in B$.

It is now evident that, for any right Gelfand quantale Q, the presheaves on Q, together with the maps of presheaves on Q, form a category

$$\textbf{Presheaves}\ /Q.$$

Moreover, from the observations that to each complete quantal set A over the quantale Q there corresponds a canonical presheaf

$$\Gamma(A)$$

on Q, and to each map $f : A \to B$ of complete quantal sets over Q a canonical map

$$\Gamma(f) : \Gamma(A) \to \Gamma(B)$$

of presheaves on Q, it may be shown straightforwardly that one has a functor

$$\Gamma : \textbf{Complete } Q\textbf{-Sets} \to \textbf{Presheaves}\ /Q,$$

which establishes the first part of the following:

Theorem. *For any right Gelfand quantale Q, there are adjoint functors*

$$\textbf{Presheaves}\ /Q \ \underset{\Gamma}{\overset{U}{\rightleftarrows}}\ \textbf{Complete } Q\textbf{-Sets}$$

of which the coadjoint assigns to each complete quantal set over Q its canonical presheaf on Q.

To establish the existence of the adjoint to the underlying presheaf functor, we shall first construct a functor

$$U : \textbf{Presheaves}\ /Q \to Q\textbf{-Sets}$$

by showing that each presheaf A on the quantale Q admits an underlying structure as a quantal set over Q, and that each map of presheaves on Q determines a map of the corresponding underlying quantal sets over Q. Taking inspiration from the case of a presheaf on a

locale, we define the equality of elements $a, b \in A$ of a presheaf A to be the element of the quantale Q given by:

$$[\![a = b]\!] = Ea \,\&\, \bigvee_{a|p=p|b} p \,\&\, Eb.$$

Intuitively, this is the join of those elements of the quantale on which the elements $a, b \in A$ are both defined and have identical restriction.

It is asserted first that, with respect to the equality

$$[\![\cdot = \cdot]\!] : A \times A \to Q$$

thereby defined, together with the extent mapping

$$E : A \to Q$$

of the presheaf, the underlying set A of the presheaf is indeed a quantal set over Q. Verifying each of the axioms in turn, we argue as follows:

- $Ea\&Ea \leq [\![a = a]\!]$ since $Ea\&Ea \leq Ea\&Ea\&Ea \leq Ea\& \bigvee_{a|p=p|a} p\,\&Ea$ because $a \upharpoonleft Ea = a = Ea \downharpoonleft a$;

- $[\![a = b]\!] \leq Ea\&Eb$ since $[\![a = b]\!] = Ea \,\&\, \bigvee_{a|p=p|b} p \,\&\, Eb \leq Ea\&Eb$ by the right-sidedness of the quantale Q;

- $Ea \,\&[\![b = a]\!] \leq [\![a = b]\!]$ since

$$\begin{aligned} Ea\,\&[\![b = a]\!] &= \textstyle\bigvee_{b|p=p|a} Ea\&\, Eb\&\, p\&\, Ea \\ &= \textstyle\bigvee_{b|p=p|a} Ea\&(Eb\&\, p)\&Eb \\ &\leq \textstyle\bigvee_{a|q=q|b} Ea \,\&\, q \,\&\, Eb \end{aligned}$$

since given any $p \in Q$ for which $b \upharpoonleft p = p \downharpoonleft a$ we have that $p \downharpoonleft a = p \downharpoonleft p \downharpoonleft a = p \downharpoonleft b \upharpoonleft p$, by the idempotency and the choice of $p \in Q$, hence $Eb\&p \downharpoonleft a = Eb\&p \downharpoonleft b \upharpoonleft p = Eb\&p \downharpoonleft b \upharpoonleft Eb\&p = E(b \upharpoonleft Eb\&p) \downharpoonleft (b \upharpoonleft Eb\&p) = b \upharpoonleft Eb\&p$; and finally

- $[\![a = b]\!]\&[\![b = c]\!] \leq [\![a = c]\!]$ since

$$\begin{aligned} [\![a = b]\!]\&[\![b = c]\!] &= Ea\& \textstyle\bigvee_{a|p=p|b} p \,\&\, Eb \,\&\, Eb\& \bigvee_{b|q=q|c} q \,\&\, Ec \\ &\leq Ea \,\&\, \textstyle\bigvee_{a|p=p|b} \bigvee_{b|q=q|c} p\&q \,\&\, Ec \\ &\leq Ea \,\&\, \textstyle\bigvee_{a|r=r|c} r \,\&\, Ec \end{aligned}$$

since $a \upharpoonleft p\&q = p\&q \downharpoonleft c$ because $a \upharpoonleft p\&q = (a \upharpoonleft p) \upharpoonleft q = (p \downharpoonleft b) \upharpoonleft q = p \downharpoonleft (b \upharpoonleft q) = p \downharpoonleft (q \downharpoonleft c) = p\&q \downharpoonleft c$, as required. The underlying set of the presheaf A is therefore a quantal set, as asserted. It may be remarked in passing that it is the particular form of this proof of the transitivity of equality that motivates the symmetry of the condition

$$a \upharpoonleft p = p \downharpoonleft b$$

appearing in its definition.

Consider now a map

$$f : A \to B$$

of presheaves on Q, and define a mapping

$$f : A \times B \to Q$$

by writing

$$f(a, b) = [\![f(a) = b]\!]$$

for each $a \in A$ and $b \in B$, where the equality relation is that of the quantal set canonically determined by the presheaf B. It is asserted that this defines a map

$$f : A \to B$$

of the quantal sets determined by the presheaves concerned.

Once again, we verify the axioms one by one, working principally from the properties of equality on the quantal set A :

- $f(a, b) \leq Ea\&Eb$ since $[\![f(a) = b]\!] \leq Ef(a)\&Eb = Ea\&Eb$;
- $[\![a = a']\!]\&f(a', b) \leq f(a, b)$ by

$$[\![a = a']\!]\&[\![f(a') = b]\!] \leq [\![f(a) = f(a')]\!]\&[\![f(a') = b]\!] \leq [\![f(a) = b]\!], \text{ since}$$
$$[\![a = a']\!] = Ea \,\&\, \bigvee_{a|p=p|a'} p \,\&\, Ea' \leq Ef(a) \,\&\, \bigvee_{f(a|p)=f(p|a')} p \,\&\, Ef(b)$$
$$= Ef(a) \,\&\, \bigvee_{f(a)|p=p|f(a')} p \,\&\, Ef(b) = [\![f(a) = f(a')]\!]$$

by the properties of a map of presheaves on Q ;

- $f(a, b)\,\&[\![b = b']\!] \leq f(a, b')$ since $[\![f(a) = b]\!]\&[\![b = b']\!] \leq [\![f(a) = b']\!]$;
- $Eb\& f(a, b)\,\& f(a, b')\,\&Eb' \leq [\![b = b']\!]$ since

$$Eb\&[\![f(a) = b]\!]\&[\![f(a) = b']\!]\& Eb' \leq [\![b = f(a)]\!]\&[\![f(a) = b']\!]\& Eb'$$
$$\leq [\![b = b']\!]\& Eb' = [\![b = b']\!] \,;$$

- $Ea \leq \bigvee_b f(a, b)$ by $Ea = Ea \,\&\, Ea = Ef(a) \,\&\, Ef(a) \leq [\![f(a) = f(a)]\!] \leq \bigvee_b [\![f(a) = b]\!]$ $= \bigvee_b f(a, b)$, which completes the verification that this yields a map of quantal sets.

It may be proved straightforwardly that this determines a functor, which we shall denote by

U : **Presheaves** /$Q \to Q$-**Sets**

from the category of presheaves on the quantale Q to the category of quantal sets over Q, of which the composite with the functor

Γ : **Complete** Q-**Sets** $\to$ **Presheaves** /Q

that assigns to each complete quantal set over Q its canonical presheaf is exactly the inclusion functor

I : **Complete** Q-**Sets** $\to Q$-**Sets**

of the category of complete quantal sets over Q in that of quantal sets over Q.

From this latter assertion, to the proof of which we now turn, the existence of the required adjoint may then be deduced from that of the completion of quantal sets over Q. It must be proved, therefore, firstly, that, for any complete quantal set A, the quantal set

$$U(\Gamma(A))$$

determined by the canonical presheaf $\Gamma(A)$ of A is exactly the quantal set A. Since the sets and the extent operators concerned are identical throughout, it is necessary only to show that the equality relations on the sets are the same.

Recalling the definition of the equality relation on the quantal set underlying a presheaf, we have to show that

$$\bigvee_{a|p=p|b} Ea\& p\& Eb = [\![a = b]\!]$$

for any elements $a, b \in A$ of the complete quantal set A. To prove this, observe firstly that

$$a \upharpoonleft [\![a = b]\!] = [\![a = b]\!] \downharpoonright b,$$

since, by the definition of the restrictions on the canonical presheaf of A, this is just the statement that $[\![a = c]\!]\&[\![a = b]\!] = [\![a = b]\!]\&[\![b = c]\!]$ for any $c \in A$. Certainly,

$$[\![a = b]\!]\&[\![b = c]\!] = [\![a = b]\!]\&[\![b = c]\!]\&[\![a = b]\!] \leq [\![a = c]\!]\&[\![a = b]\!],$$

while conversely,

$$\begin{aligned}[\![a = c]\!]\&[\![a = b]\!] &= Ea\&[\![a = c]\!]\&[\![a = b]\!] \\ &= Ea\&[\![a = b]\!]\&[\![a = c]\!] \\ &= Ea\&[\![a = b]\!]\&[\![b = a]\!]\&[\![a = c]\!] \\ &\leq Ea\&[\![a = b]\!]\&[\![b = c]\!] \\ &= [\![a = b]\!]\&[\![b = c]\!],\end{aligned}$$

establishing the required identity. Hence, observing that $[\![a = b]\!] = Ea\&[\![a = b]\!]\&Eb$, one has that

$$[\![a = b]\!] \leq \bigvee_{a|p=p|b} Ea\& p\& Eb.$$

To establish the converse, it must be shown that

$$Ea\& p\& Eb \leq [\![a = b]\!]$$

for each $p \in Q$ satisfying $a \upharpoonleft p = p \downharpoonright b$. Observe firstly that $a \upharpoonleft p = p \downharpoonright b$ means that for all $c \in A$ one has that $[\![a = c]\!]\&p = p\&[\![b = c]\!]$. Hence, in particular, $Ea\&p = p\&[\![b = a]\!] = p\&$ $[\![a = b]\!]$, giving that $Ea\&p\&Eb = Ea\&Ea\&p\&Eb = Ea\&p\&[\![a = b]\!]\&Eb \leq Ea\&[\![a = b]\!]\&Eb =$ $[\![a = b]\!]$, as required. Thus,

$$\bigvee_{a|p=p|b} Ea\& p\& Eb = [\![a = b]\!],$$

as asserted.

Consider now a map $f: A \to B$ of complete quantal sets over Q, given by a functional relation

$$f: A \times B \to Q.$$

Observe that the map $\Gamma(f): \Gamma(A) \to \Gamma(B)$ of canonical presheaves assigns to each $a \in A$ the element $f(a) \in \Gamma(B)$ uniquely satisfying

$$[\![f(a) = b]\!] = f(a,b)$$

for each $b \in B$. Applying the functor U which gives each presheaf its underlying structure as a quantal set, we have that

$$U\Gamma(f) : U\Gamma(A) \to U\Gamma(B)$$

is given by the functional relation that assigns to each pair $(a,b) \in U\Gamma(A) \times U\Gamma(B)$ the element $[\![f(a) = b]\!] \in Q$ given by the equality relation on the quantal set underlying the presheaf $\Gamma(B)$. By the above remarks, this is exactly the element $f(a,b) \in Q$, yielding that map of canonical quantal sets thereby determined is exactly $I(f) : I(A) \to I(B)$. Hence, one has that the functors $U\Gamma$ and I are identical, as asserted.

We now apply this to show that the functors

$$\textbf{Presheaves } /Q \underset{\Gamma}{\overset{\Sigma}{\rightleftarrows}} \textbf{Complete } Q\textbf{-Sets}$$

given respectively by SU and Γ are adjoint. This will be done by describing the adjunction and coadjunction maps

$$\tau : \textbf{Presheaves } /Q \to \Gamma\Sigma \text{ and } \sigma : \Sigma\Gamma \to \textbf{Complete } Q\textbf{-Sets}$$

and verifying that the required adjunction identities are satisfied. Observe firstly that, by the above remarks, the functor $\Sigma\,\Gamma$ is exactly the functor SI assigning to each complete quantal set the completion of its underlying quantal set. The coadjunction σ may therefore be defined to be the coadjunction

$$\varepsilon : SI \to \textbf{Complete } Q\textbf{-Sets}$$

of the adjointness given by completion of quantal sets over Q. For the adjunction, we define τ by taking for each presheaf A on Q the map

$$\tau_A : A \to \Gamma SU(A)$$

that assigns to each $a \in A$ the element $\tilde{a} \in SU(A)$ of the completion of the quantal set canonically determined by the presheaf A. It may be verified straightforwardly that this is indeed a map of presheaves on Q and that this yields a natural map of the required functors.

It remains to show that the maps

$$\Sigma(A) \xrightarrow{\Sigma(\tau_A)} \Sigma\Gamma\Sigma(A) \xrightarrow{\sigma_{\Sigma(A)}} \Sigma(A) \text{ and } \Gamma(B) \xrightarrow{\tau_{\Gamma(B)}} \Gamma\Sigma\Gamma(B) \xrightarrow{\Gamma(\sigma_B)} \Gamma(B)$$

are the respective identities for each presheaf A on Q and for each for each $a \in A$ and each $s \in \Gamma\Sigma(A)$, where the equality relation is that of the quantal set underlying the presheaf $\Gamma\Sigma(A)$. But this quantal set $U\Gamma\Sigma(A)$ is exactly $I\Sigma(A)$, by the remarks made earlier, and hence its equality relation is exactly that of the complete quantal set B over Q. Concerning the natural map τ, it may be remarked firstly that

$$U(\tau_A) = \eta_{U(A)}$$

for any presheaf A, since $\tau_A(a)$ is defined to be the singleton $\tilde{a}$, and hence

$$U(\tau_A)(a,s) = [\![\tau_A(a) = s]\!],$$

completion of the quantal set $U(A)$ canonically determined by the presheaf A. Hence,

$$U(\tau_A)(a,s) = [\![\tilde{a} = s]\!] = \eta_{U(A)}(a,s)$$

for each $a \in A$ and each $s \in \Gamma\Sigma(A)$, by the definition of the natural map η : Q-**Sets** $\to$ ISdetermined by completion of quantal sets over Q.

Recalling that the functor Σ is defined to be the composite SU, and that the natural map σ is exactly the coadjunction:

$$\varepsilon : SI \to \textbf{Complete } Q\textbf{-Sets},$$

we observe that $\Sigma(\tau_A) = SU(\tau_A) = S\eta_{U(A)}$, and $\sigma_{\Sigma(A)} = \eta_{SU(A)}$, from which it follows that

$$\Sigma(A) \xrightarrow{\Sigma(\tau_A)} \Sigma\Gamma\Sigma(A) \xrightarrow{\sigma_{\Sigma(A)}} \Sigma(A)$$

is the identity on $\Sigma(A)$, by the corresponding identity between η and ε at the quantal set $U(A)$.

Now consider the other required identity, namely that

$$\Gamma(B) \xrightarrow{\tau_{\Gamma(B)}} \Gamma\Sigma\Gamma(B) \xrightarrow{\Gamma(\sigma_B)} \Gamma(B)$$

is the identity on $\Gamma(B)$ for any complete quantal set B. Observe that $\tau_{\Gamma(B)}$ assigns to each element $a \in \Gamma(B)$ of the canonical presheaf of B the element $\tilde{a} \in \Gamma SU\Gamma(B)$ of the canonical presheaf of the completion of the quantal set $U\Gamma(B)$. Applying again the fact that $U\Gamma$ is exactly I, and observing that $\varepsilon_B : SI(B) \to B$ is given by assigning to each $s \in SI(B)$ and $b \in B$ the element $s(b) \in Q$, we have that

$$\sigma_B(\tilde{a}, b) = [\![a = b]\!].$$

Hence, $\Gamma(\sigma_B)(\tilde{a})$ is exactly $a \in B$, since this is the unique element of B satisfying the above condition. The composite is therefore the identity on the underlying presheaf of the complete quantal set B, as required, from which it follows that the functors

$$\textbf{Presheaves } /Q \underset{\Gamma}{\overset{\Sigma}{\rightleftarrows}} \textbf{Complete } Q\textbf{-Sets}$$

are indeed adjoint, as asserted. $\Box$

We have in fact proved more than simply that these functors are adjoint. For, recalling that the coadjunction is exactly the coadjunction

$$\varepsilon : SI \to \textbf{Complete } Q\textbf{-Sets}$$

determined by the completion of quantal sets over Q, which has already been shown to be a natural isomorphism, we obtain the following:

COROLLARY. *For any right Gelfand quantale Q, the adjoint functors*

$$\textbf{Presheaves } /Q \underset{\Gamma}{\overset{\Sigma}{\rightleftarrows}} \textbf{Complete } Q\textbf{-Sets}$$

of which the adjoint assigns to each presheaf its completion in the category of quantal sets provide a reflection of the category of presheaves on Q into the category of complete quantal sets over Q. $\Box$

With this adjointness indicating firmly that the canonical presheaves of complete quantal sets are exactly the sheaves on the quantale Q, we may now proceed to define the concept of a sheaf on the quantale Q. Before doing so, however, there is an important observation that may be made concerning the left and right restrictions on a presheaf A, namely, that, by the idempotency and right-sidedness of the quantale Q, one has that:

$$a \upharpoonleft p \;=\; Ea \,\&\, p \downharpoonright a$$

for any $a \in A$ and any $p \in Q$. In particular, the right restriction on any presheaf A may be defined in terms of the left restriction.

To see this, we remark that

$$a \upharpoonleft p \;=\; E(a \upharpoonleft p) \downharpoonright (a \upharpoonleft p),$$

since left (or right) restriction to the extent of an element leaves it unchanged. Then,

$$E(a \upharpoonleft p) \downharpoonright (a \upharpoonleft p) \;=\; Ea \,\&\, p \downharpoonright (a \upharpoonleft Ea \,\&\, p \,\&\, Ea),$$

since $a \upharpoonleft p = (a \upharpoonleft Ea) \upharpoonleft p = a \upharpoonleft Ea \,\&\, p = a \upharpoonleft Ea \,\&\, Ea \,\&\, p = a \upharpoonleft Ea \,\&\, p \,\&\, Ea$ by the idempotency and right-sidedness of the quantale. But,

$$Ea \,\&\, p \downharpoonright (a \upharpoonleft Ea \,\&\, p \,\&\, Ea) = (Ea \,\&\, p \downharpoonright a) \upharpoonleft E(Ea \,\&\, p \downharpoonright a),$$

by the distributivity of left and right restriction, and

$$(Ea \,\&\, p \downharpoonright a) \upharpoonleft E(Ea \,\&\, p \downharpoonright a) = Ea \,\&\, p \downharpoonright a\ ,$$

by right restriction to the extent being the identity. Hence,

$$a \upharpoonleft p \;=\; Ea \,\&\, p \downharpoonright a,$$

as asserted.

Although it may straightforwardly be shown that the concept of presheaf already introduced is indeed equivalent to one expressed simply in terms of left restriction, we have chosen to retain both left and right restriction because the concept of sheaf may be defined most naturally in that context, to which we now proceed.

5. SHEAVES ON Q

With the notion of presheaf established, together with its connection to that of quantal set, the identification of a concept of sheaf on a quantale comes down to describing a sheaf condition on a presheaf, corresponding to that of the completeness of a quantal set. The expectation is that this will extend that known in the case of local sets, and that the existing embedding of the category of complete quantal sets in the category of presheaves on the quantale will provide an isomomorphism with the category of sheaves on the quantale. Further, it may be expected that the idea of a compatible family of elements of a presheaf will be identifiable in some sense with that of a singleton subset of a quantal set. The existence of a unique element obtained by patching a given compatible family will then correspond to the completeness of the quantal set.

Consider first a complete quantal set A, together with its canonical presheaf $\Gamma(A)$. For any singleton subset $s : A \to Q$, consider the family

$$(\, s(a) \downharpoonright a\,)_{a \in A}$$

of elements of the presheaf $\Gamma(A)$ indexed by the quantal set A. Then we assert that the elements of this family satisfy the condition that

$$(s(a) \downharpoonright a) \upharpoonleft s(b) \;=\; s(a) \downharpoonright (s(b) \downharpoonright b)$$

for any $a, b \in A$.

For, to show these restrictions are equal, it is sufficient to show that, for any $c \in A$, one has

$$[\![(s(a) \downharpoonright a) \upharpoonleft s(b) = c]\!] = [\![s(a) \downharpoonright (s(b) \downharpoonright a) = c]\!]$$

in the quantal set A. But, by the definitions of left and right restriction in a complete quantal set, one has that

$$[\![(s(a) \downharpoonright a) \upharpoonleft s(b) = c]\!] = s(a) \,\&\, [\![a = c]\!] \,\&\, s(b),$$

and $$[\![s(a) \downharpoonright (s(b) \downharpoonright b) = c]\!] = s(a) \,\&\, s(b) \,\&\, [\![b = c]\!] = s(a) \,\&\, [\![b = c]\!] \,\&\, s(b),$$

by the right symmetricity of the quantale. The required equality follows by remarking that, on the one hand,

$$\begin{aligned} s(a) \,\&\, [\![a = c]\!] \,\&\, s(b) &= s(a) \,\&\, s(a) \,\&\, [\![a = c]\!] \,\&\, s(b) \\ &= s(a) \,\&\, s(c) \,\&\, s(b) \\ &= s(a) \,\&\, s(c) \,\&\, s(b) \,\&\, s(b) \,\&\, Eb \end{aligned}$$

$$= s(a) \mathbin{\&} Eb \mathbin{\&} s(b) \mathbin{\&} s(c) \mathbin{\&} s(b)$$

$$\leq s(a) \mathbin{\&} [\![b = c]\!] \mathbin{\&} s(b),$$

by applying the properties of a singleton subset, together with the right symmetricity and idempotency of the quantale; while, conversely, we have that

$$\begin{aligned} s(a) \mathbin{\&} [\![b = c]\!] \mathbin{\&} s(b) &= s(a) \mathbin{\&} s(a) \mathbin{\&} [\![b = c]\!] \mathbin{\&} s(b) \mathbin{\&} s(b) \\ &= s(a) \mathbin{\&} Ea \mathbin{\&} s(a) \mathbin{\&} s(b) \mathbin{\&} [\![b = c]\!] \mathbin{\&} s(b) \\ &\leq s(a) \mathbin{\&} [\![a = b]\!] \mathbin{\&} [\![b = c]\!] \mathbin{\&} s(b) \\ &\leq s(a) \mathbin{\&} [\![a = c]\!] \mathbin{\&} s(b), \end{aligned}$$

by similar arguments.

Denoting by

$$a_s \in A$$

the element $s(a) \downharpoonright a$ obtained by localising $a \in A$ to the extent to which it lies in the subset $s \in S(A)$, and observing that

$$E(s(a) \downharpoonright a) \;=\; s(a) \mathbin{\&} Ea \;=\; s(a),$$

we may express the condition satisfied by the family $(a_s)_{a \in A}$ determined by the singleton subset $s \in S(A)$ by writing that:

$$a_s \upharpoonright Eb_s \;=\; Ea_s \downharpoonright b_s$$

for each $a, b \in A$. The fact that the family of elements is derived from a singleton subset of the complete quantal set therefore implies a certain compatibility between the elements chosen in the underlying presheaf.

It may be remarked that these observations in fact have a converse. One may, of course, obtain such a family

$$(a_s)_{a \in A}$$

of elements of the underlying presheaf $\Gamma(A)$ of a complete quantal set A by localising at any subset

$$s : A \to Q$$

of the quantal set A. The converse assertion is that, if the family so obtained satisfies the compatibility condition expressed above, then the subset is indeed a singleton subset of A. For, given any $a, b \in A$, one has that:

$$\begin{aligned} Ea \mathbin{\&} s(a) \mathbin{\&} s(b) &= Ea \mathbin{\&} s(a) \mathbin{\&} s(b) \mathbin{\&} [\![b = b]\!] \\ &= Ea \mathbin{\&} [\![s(a) \downharpoonright (s(b) \downharpoonright b) \;=\; b]\!] \\ &= Ea \mathbin{\&} [\![(s(a) \downharpoonright a) \upharpoonright s(b) \;=\; b]\!] \\ &= Ea \mathbin{\&} s(a) \mathbin{\&} [\![a = b]\!] \mathbin{\&} s(b) \\ &\leq Ea \mathbin{\&} [\![a = b]\!] \end{aligned}$$

$$= [\![a = b]\!].$$

With these observations, we may now make the following:

DEFINITION. For any presheaf A on the right Gelfand quantale Q, a family

$$(a_i)_{i \in I}$$

of elements of A will be said to be *compatible* provided that

for each $i, j \in I$.

In these terms, we have shown above that in the underlying presheaf $\Gamma(A)$ of a complete quantal set A, the family $(s(a) \downharpoonright a)_{a \in A}$ obtained by localising the elements of $\Gamma(A)$ at a singleton subset $s \in S(A)$ is compatible. Moreover, we have shown that this compatibility characterises the property of a subset being a singleton. It may further be remarked that the completeness of the quantal set implies the existence of a unique element $b \in \Gamma(A)$ for which

$$Ea_s \downharpoonright b \;=\; a_s$$

for each $a \in A$, and

$$Eb \;=\; \bigvee\nolimits_{a \in A} Ea_s\,.$$

For, taking for $b \in \Gamma(A)$ the unique element representing the singleton subset $s \in S(A)$, in the sense that $s(a) = [\![b = a]\!]$ for each $a \in A$, to show that $Ea_s \downharpoonright b = a_s$ for any $a \in A$, it suffices to show that

$$[\![Ea_s \downharpoonright b \;=\; c]\!] = [\![a_s \;=\; c]\!]$$

for each $c \in A$. But, on the one hand,

$$[\![Ea_s \downharpoonright b \;=\; c]\!] = Ea_s \mathbin{\&} [\![b = c]\!] = s(a) \mathbin{\&} [\![b = c]\!] = [\![b = a]\!] \mathbin{\&} [\![b = c]\!],$$

while, on the other,

$$[\![a_s \;=\; c]\!] = [\![s(a) \downharpoonright a \;=\; c]\!] = s(a) \mathbin{\&} [\![a \;=\; c]\!] = [\![b \;=\; a]\!] \mathbin{\&} [\![a \;=\; c]\!].$$

However, $$[\![b = a]\!] \mathbin{\&} [\![b = c]\!] = [\![b = a]\!] \mathbin{\&} [\![b = a]\!] \mathbin{\&} [\![b = c]\!]$$

$$= [\![b = a]\!] \mathbin{\&} [\![a = b]\!] \mathbin{\&} [\![b = c]\!] \leq [\![b = a]\!] \mathbin{\&} [\![a = c]\!],$$

while, conversely,

$$[\![b = a]\!] \mathbin{\&} [\![a = c]\!] = [\![b = a]\!] \mathbin{\&} [\![b = a]\!] \mathbin{\&} [\![a = c]\!] \leq [\![b = a]\!] \mathbin{\&} [\![b = c]\!],$$

giving the required equality. Moreover, observing that $Ea_s = s(a)$ and that $b_s = s(b) \downharpoonright b$, so that $Eb_s = s(b) = [\![b = b]\!] = Eb$, we have that $Ea_s = s(a) = [\![b = a]\!] \leq Eb \mathbin{\&} Ea \leq Eb$ for each $a \in A$, and hence that

$$Eb \;=\; \bigvee\nolimits_{a \in A} Ea_s,$$

as asserted. Hence, the element $b \in A$ has the required properties. Moreover, it is unique: since if $b' \in A$ also has these properties, then necessarily

$$Eb = \bigvee_{a \in A} Ea_s = Eb',$$

from which it follows that

$$b = Eb \downharpoonright b = s(b) \downharpoonright b = b_s = Eb_s \downharpoonright b' = Eb \downharpoonright b' = Eb' \downharpoonright b' = b',$$

as required.

Again, with these observations in mind, we make the following:

Definition. By a *sheaf* on a right Gelfand quantale Q will be meant a presheaf A on Q for which given any family

$$(a_i)_{i \in I}$$

of elements of A that is compatible there exists a unique element $a \in A$ for which

$$Ea_i \downharpoonright a = a_i \quad \text{for each } i \in I,$$

and of which the extent Ea is equal to $\bigvee_{i \in I} Ea_i$.

The element $a \in A$ obtained in this way will be said to be the *join* of the compatible family.

At this point, we have shown only that the compatible family

$$(a_s)_{a \in A}$$

determined by localisation at any singleton subset $s \in S(A)$ of a complete quantal set A has a unique join in its canonical presheaf $\Gamma(A)$. However, this may be extended to the case of an arbitrary compatible family

$$(a_i)_{i \in I}$$

of elements of the underlying presheaf $\Gamma(A)$, by considering the singleton subset $s : A \to Q$ of A defined by writing

$$s(a) = \bigvee_{i \in I} [\![a_i = a]\!]$$

for each $a \in A$. That this defines a subset of A holds irrespective of the compatibility of the family: for

$$\begin{aligned} s(a) \,\&\, [\![a = a']\!] &= \bigvee_{i \in I} [\![a_i = a]\!] \,\&\, [\![a = a']\!] \\ &\le \bigvee_{i \in I} [\![a_i = a']\!] \\ &= s(a'), \end{aligned}$$

for any $a, a' \in A$, and

$$s(a) = \bigvee_{i \in I} [\![a_i = a]\!] \le \bigvee_{i \in I} [\![a_i = a]\!] \,\&\, Ea = s(a) \,\&\, Ea$$

for any $a \in A$. In the case of compatibility, the subset generated by the family is a singleton: since, for any $a, b \in A$ and $i, j \in I$, we have that

$$\begin{aligned} Ea \,\&\, [\![a_i = a]\!] \,\&\, [\![a_j = b]\!] &= Ea \,\&\, Ea_i \,\&\, [\![a_i = a]\!] \,\&\, Ea_j \,\&\, [\![a_j = b]\!] \\ &= Ea \,\&\, [\![a_i = a]\!] \,\&\, Ea_j \,\&\, Ea_i \,\&\, [\![a_j = b]\!] \\ &= Ea \,\&\, [\![a_i \upharpoonright Ea_j = a]\!] \,\&\, [\![Ea_i \downharpoonright a_j = b]\!] \\ &= Ea \,\&\, [\![a = a_i \upharpoonright Ea_j]\!] \,\&\, [\![Ea_i \downharpoonright a_j = b]\!] \\ &\le Ea \,\&\, [\![a = b]\!] \\ &= [\![a = b]\!], \end{aligned}$$

since $a_i \upharpoonright Ea_j = Ea_i \downharpoonright a_j$. From this, it follows that

$$Ea \,\&\, s(a) \,\&\, s(b) \le Ea \,\&\, \bigvee_{i \in I} [\![a_i = a]\!] \,\&\, \bigvee_{j \in I} [\![a_j = b]\!] \le [\![a = b]\!],$$

as asserted.

By the completeness of the quantal set A, let $a \in A$ denote the unique element for which

$$s(b) = [\![a = b]\!]$$

for each $b \in A$. It is asserted that the element $a \in \Gamma(A)$ is the unique join of the compatible family $(a_i)_{i \in I}$ of elements of the underlying presheaf $\Gamma(A)$. For, firstly,

$$Ea = \bigvee_{i \in I} Ea_i,$$

since, on the one hand,

$$Ea_i \le \bigvee_{j \in I} [\![a_j = a_i]\!] = s(a_i) = [\![a = a_i]\!] \le Ea$$

for each $i \in I$, while, on the other,

$$Ea = [\![a = a]\!] = s(a) = \bigvee_{i \in I} [\![a_i = a]\!] \le \bigvee_{i \in I} Ea_i.$$

Moreover, for any $c \in A$, we have that:

$$\begin{aligned} [\![Ea_i \downharpoonright a = c]\!] &= Ea_i \,\&\, [\![a = c]\!] = Ea_i \,\&\, s(c) = Ea_i \,\&\, \bigvee_{j \in I} [\![a_j = c]\!] \\ &= \bigvee_{j \in I} Ea_i \,\&\, [\![a_j = c]\!] = \bigvee_{j \in I} [\![Ea_i \downharpoonright a_j = c]\!] \\ &= \bigvee_{j \in I} [\![a_i \upharpoonright Ea_j = c]\!] = \bigvee_{j \in I} [\![a_i = c]\!] \,\&\, Ea_j \\ &\le [\![a_i = c]\!], \end{aligned}$$

while conversely,

$$\begin{aligned} [\![a_i = c]\!] &= Ea_i \,\&\, [\![a_i = c]\!] \le Ea_i \,\&\, \bigvee_{j \in I} [\![a_j = c]\!] = Ea_i \,\&\, s(c) \\ &= Ea_i \,\&\, [\![a = c]\!] = [\![Ea_i \downharpoonright a = c]\!]. \end{aligned}$$

Hence,

$$[\![Ea_i \downharpoonright a = c]\!] = [\![a_i = c]\!]$$

for any $c \in A$, from which it follows that:

$$Ea_i \downharpoonright a = a_i$$

for each $i \in I$.

Finally, the uniqueness of this join follows by observing that any $a \in \Gamma(A)$ satisfying these conditions necessarily represents the singleton subset generated by the compatible family $(a_i)_{i\in I}$, since, for any $c \in A$, we have that:

$$\begin{aligned} [\![a = c]\!] &= Ea \,\&\, [\![a = c]\!] \\ &= \textstyle\bigvee_{i\in I} Ea_i \,\&\, [\![a = c]\!] \\ &= \textstyle\bigvee_{i\in I} [\![Ea_i \downharpoonright a = c]\!] \\ &= \textstyle\bigvee_{i\in I} [\![a_i = c]\!] \\ &= s(c). \end{aligned}$$

Hence, by the uniqueness of the element $a \in A$ representing the singleton subset $s \in S(A)$ determined by the compatible family $(a_i)_{i\in I}$ in the presheaf $\Gamma(A)$, the join of the family is unique. In particular, the canonical presheaf $\Gamma(A)$ of any complete quantal set A is necessarily a sheaf on the quantale Q.

With these remarks, denoting for any right Gelfand quantale Q by

Sheaves $/Q$

the full subcategory of the category **Presheaves** $/Q$ of presheaves on Q determined by the sheaves, we see that the functor

Γ : **Complete** Q**-Sets** → **Presheaves** $/Q$

that assigns to each complete quantal set over Q its canonical presheaf may in fact be considered as a functor into the category of sheaves on Q , concerning which we have the following:

THEOREM. *For any right Gelfand quantale Q, the categories of complete quantal sets over Q and of sheaves on Q are isomorphic by the functors*

$$\textbf{Complete } Q\textbf{-Sets} \underset{U}{\overset{\Gamma}{\rightleftarrows}} \textbf{Sheaves } /Q$$

that assign respectively to each complete quantal set its canonical presheaf, and to each sheaf its underlying structure as a quantal set.

The existence of the functor from **Complete** Q**-Sets** to **Sheaves** $/Q$ has already been remarked. The inverse functor from **Sheaves** $/Q$ to **Complete** Q**-Sets** is obtained by establishing that the quantal set $U(A)$ canonically obtained from a sheaf A by taking the extent of any element to be that in the sheaf A, and by considering the equality relation given by writing

$$[\![a = b]\!] = \textstyle\bigvee_{a|p=p|b} Ea \,\&\, p \,\&\, Eb$$

for any elements $a, b \in A$, is actually complete.

Before proving this, we make an observation relating this equality relation defined on the underlying set of elements of the sheaf A to its structure as a presheaf on the quantale Q, namely that

$$a \upharpoonleft [\![a = b]\!] = [\![a = b]\!] \downharpoonright b,$$

for any elements $a, b \in A$ of the presheaf A. It is in this sense that the equality of elements of a sheaf behaves like that familiar in the case of a locale, thereby justifying the definition of the equality relation considered. To verify this assertion in the case of a sheaf A, observe firstly that

$$a \upharpoonleft p = p \downharpoonright b$$

implies that $a \upharpoonleft Ea \,\&\, p \,\&\, Eb = Ea \,\&\, p \,\&\, Eb \downharpoonright b$, since $a \upharpoonleft Ea \,\&\, p \,\&\, Eb = a \upharpoonleft p \,\&\, Eb$ $= p \downharpoonright b \upharpoonleft Eb = p \downharpoonright b$, and $Ea \,\&\, p \,\&\, Eb \downharpoonright b = Ea \,\&\, p \downharpoonright b = Ea \downharpoonright a \upharpoonleft p = a \upharpoonleft p$, by the remarks concerning the relationship between left and right restriction. Hence, the families

$$(a \upharpoonleft Ea \,\&\, p \,\&\, Eb)_{a|p=p|b} \quad \text{and} \quad (Ea \,\&\, p \,\&\, Eb \downharpoonright b)_{a|p=p|b}$$

of elements of the sheaf A are identical. However, $E(a \upharpoonleft Ea \,\&\, p \,\&\, Eb) = Ea \,\&\, p \,\&\, Eb$ implies that

$$\textstyle\bigvee_{a|p=p|b} E(a \upharpoonleft Ea \,\&\, p \,\&\, Eb) = [\![a = b]\!] = E(a \upharpoonleft [\![a = b]\!]),$$

while $Ea \,\&\, p \,\&\, Eb \downharpoonright (a \upharpoonleft [\![a = b]\!]) = a \upharpoonleft Ea \,\&\, p \,\&\, Eb \,\&\, [\![a = b]\!] = a \upharpoonleft Ea \,\&\, p \,\&\, Eb$, again by the relationship between left and right restrictions, and the observation that $Ea \,\&\, p \,\&\, Eb$ $\leq [\![a = b]\!]$, implies that $E(a \upharpoonleft Ea \,\&\, p \,\&\, Eb) \downharpoonright (a \upharpoonleft [\![a = b]\!]) = a \upharpoonleft Ea \,\&\, p \,\&\, Eb$ for any $p \in Q$ for which $a \upharpoonleft p = p \downharpoonright b$. Hence,

$$a \upharpoonleft [\![a = b]\!]$$

is the unique element of the sheaf A which is the join of the compatible family

$$(a \upharpoonleft Ea \,\&\, p \,\&\, Eb)_{a|p=p|b}$$

in the sheaf A. By similar remarks, we have that

$$[\![a = b]\!] \downharpoonright b$$

is the unique join of the compatible family

$$(Ea \,\&\, p \,\&\, Eb \downharpoonright b)_{a|p=p|b}$$

in the sheaf A. Since the families have already been remarked to be identical, it follows that

$$a \upharpoonleft [\![a = b]\!] = [\![a = b]\!] \downharpoonright b$$

for any $a, b \in A$, as asserted.

Now consider this quantal set $U(A)$ underlying the sheaf A, and let $s \in S(U(A))$ be any singleton subset of $U(A)$. Then the elements $(a_s)_{a\in A}$ of the sheaf A, obtained by localising the elements of A at the singleton subset by defining

$$a_s = s(a) \downharpoonleft a$$

for each $a \in A$, form a compatible family of elements of the sheaf A. For, given any elements $a, b \in A$, we observe first that

$$s(a) \,\&\, s(b) = s(a) \,\&\, s(b) \,\&\, [\![a = b]\!],$$

since $Ea \,\&\, s(a) \,\&\, s(b) \le [\![a = b]\!]$ by the singleton axiom, hence $Ea \,\&\, s(a) \,\&\, s(b) = Ea \,\&\, s(a) \,\&\, s(b) \,\&\, [\![a = b]\!]$ which gives the required equality on multiplying on the left by $s(a)$. Then,

$$\begin{aligned}
&(s(a) \downharpoonleft a) \upharpoonright s(b) \\
&= s(a) \,\&\, Ea \,\&\, s(b) \downharpoonleft a = s(a) \,\&\, s(b) \downharpoonleft a \\
&\qquad = s(a) \,\&\, s(b) \,\&\, [\![a = b]\!] \downharpoonleft a = s(a) \,\&\, s(b) \downharpoonleft (a \upharpoonright [\![a = b]\!]) \\
&\qquad = s(a) \,\&\, s(b) \downharpoonleft ([\![a = b]\!] \downharpoonleft b) = s(a) \,\&\, s(b) \,\&\, [\![a = b]\!] \downharpoonleft b \\
&\qquad = s(a) \,\&\, s(b) \downharpoonleft b = s(a) \downharpoonleft (s(b) \downharpoonleft b),
\end{aligned}$$

hence,

$$a_s \upharpoonright Eb_s = Ea_s \downharpoonleft b_s.$$

Letting $a \in A$ denote the join of this family in the sheaf A, we assert that the corresponding element $a \in U(A)$ of the quantal set determined by A represents the singleton subset $s \in S(U(A))$. Recalling that $Eb_s = s(b)$ for each $b \in A$, the fact that $a \in A$ is the join of the family implies that

$$Ea = \bigvee_{b \in A} s(b),$$

and that

$$s(b) \downharpoonleft a = s(b) \downharpoonleft b$$

for each $b \in A$. From these, we must deduce that

$$s(b) = [\![a = b]\!]$$

for each $b \in U(A)$.

In one direction, consider the element $Ea \,\&\, s(b) \downharpoonleft a$, and observe that

$$\begin{aligned}
Ea \,\&\, s(b) \downharpoonleft a &= (Ea \,\&\, s(b) \downharpoonleft a) \upharpoonright Ea \,\&\, s(b) = Ea \,\&\, s(b) \downharpoonleft (a \upharpoonright Ea \,\&\, s(b)) \\
&= a \upharpoonright Ea \,\&\, s(b),
\end{aligned}$$

respectively by right restriction by the extent, the associativity of left and right restrictions, and by left restriction by the extent, since $E(Ea \,\&\, s(b) \downharpoonleft a) = Ea \,\&\, s(b) \,\&\, Ea = Ea \,\&\, s(b)$, and $E(a \upharpoonright Ea \,\&\, s(b)) = Ea \,\&\, Ea \,\&\, s(b) = Ea \,\&\, s(b)$. Hence, since $s(b) \downharpoonleft a = s(b) \downharpoonleft b$ and $s(b) = s(b) \,\&\, Eb$, we have that $a \upharpoonright Ea \,\&\, s(b) \,\&\, Eb = Ea \,\&\, s(b) \,\&\, Eb \downharpoonleft b$. Hence,

$$Ea \,\&\, s(b) \,\&\, Eb \le [\![a = b]\!],$$

by the definition of equality in the quantal set $U(A)$. However, since $s(b) \le Ea$ and $s(b) = s(b) \,\&\, Eb$, we have that $s(b) \le Ea \,\&\, s(b) \,\&\, Eb$, from which it follows that

$$s(b) \le [\![a = b]\!].$$

In the other direction, we observe firstly that $s(b) \le s(a)$ for each $b \in U(A)$. For, by the inequality already shown, we have that $s(b) = s(b) \,\&\, s(b) \le s(b) \,\&\, [\![a = b]\!] \le s(a)$. Hence, $Ea \le s(a)$, since $Ea = \bigvee_{b \in A} s(b) \le s(a)$. Hence,

$$[\![a = b]\!] = Ea \,\&\, [\![a = b]\!] \le s(a) \,\&\, [\![a = b]\!] \le s(b),$$

from which it follows that

$$s(b) = [\![a = b]\!]$$

for each $b \in U(A)$, as required. Hence, the singleton subset $s \in S(U(A))$ is representable by $a \in U(A)$, as asserted.

The uniqueness of the element $a \in U(A)$ follows from that of the join of any compatible family in the sheaf A by observing that if $a \in U(A)$ satisfies the condition that

$$s(b) = [\![a = b]\!]$$

for all $b \in U(A)$, then necessarily

$$\bigvee_{b \in A} s(b) = Ea \quad \text{and} \quad s(b) \downharpoonleft a = s(b) \downharpoonleft b$$

for all $b \in A$. For firstly for any $b \in A$ we have that $s(b) = s(b) \,\&\, s(b) = s(b) \,\&\, [\![a = b]\!] = s(b) \& [\![b = a]\!] \le s(a) = [\![a = a]\!] = Ea$. Hence,

$$\bigvee_{b \in A} s(b) \le Ea.$$

While conversely, $Ea = [\![a = a]\!] = s(a)$; hence,

$$Ea \le \bigvee_{b \in A} s(b),$$

giving the first equality. While, for the second, since we have remarked already that, for any elements $a, b \in A$ of the sheaf, it is the case that $a \upharpoonright [\![a = b]\!] = [\![a = b]\!] \downharpoonleft b$, it follows that for any $b \in A$ we have that $[\![a = b]\!] \downharpoonleft a = [\![a = b]\!] \downharpoonleft b$, since

$$\begin{aligned}
a \upharpoonright [\![a = b]\!] &= E(a \upharpoonright [\![a = b]\!]) \downharpoonleft (a \upharpoonright [\![a = b]\!]) \\
&= Ea \,\&\, [\![a = b]\!] \downharpoonleft (a \upharpoonright [\![a = b]\!]) \\
&= (Ea \,\&\, [\![a = b]\!] \downharpoonleft a) \upharpoonright [\![a = b]\!] \\
&= (Ea \,\&\, [\![a = b]\!] \downharpoonleft a) \upharpoonright Ea \,\&\, [\![a = b]\!] \& Ea \\
&= Ea \,\&\, [\![a = b]\!] \downharpoonleft a \\
&= [\![a = b]\!] \downharpoonleft a.
\end{aligned}$$

It follows that

$$s(b) \downharpoonleft a = s(b) \downharpoonleft b$$

for any $b \in A$. Hence, the element $a \in A$ is the unique join of the family $(b_s)_{b \in A}$ obtained by localising at the singleton subset $s \in S(U(A))$. Hence, the singleton subset is thus uniquely representable by the element $a \in A$, establishing the completeness of the quantal set $U(A)$.

Having established the existence of these functors

$$\textbf{Complete } Q\textbf{-Sets} \underset{U}{\overset{\Gamma}{\rightleftarrows}} \textbf{Sheaves } /Q,$$

it will now be shown that they are mutually inverse. In one direction, the work has already been done. Within the broader context of the category **Presheaves** /Q, it was shown that the composite

Complete Q**-Sets** → **Presheaves** /Q → Q**-Sets**

was indeed the inclusion functor of the full subcategory of complete quantal sets in the category of quantal sets over Q. In the present context, we therefore have that $U\Gamma$ is the identity functor on the category **Complete** Q**-Sets**.

Consider then the composite

Sheaves /Q → **Complete** Q**-Sets** → **Sheaves** /Q

which assigns to each sheaf A the presheaf $\Gamma U(A)$ canonically determined by its underlying quantal set $U(A)$. Since the underlying sets and the extent operators of each of these structures are identical, to show that $\Gamma U(A)$ is exactly A is equivalent to showing that its restrictions are exactly those of the sheaf A. By the concluding observations of the preceding section, it suffices to show that the left restrictions coincide.

Suppose then that A is a sheaf on the quantale Q. For any element $a \in A$, it is asserted that the left restriction $q \downharpoonleft a \in A$ at any element $q \in Q$ is equal to that of the corresponding element of the sheaf $\Gamma U(A)$. By the construction of the left restriction on the canonical presheaf of a complete quantal set, this assertion is equivalent to showing that $q \downharpoonleft a \in A$ represents the singleton subset of $U(A)$ obtained by assigning to each $c \in U(A)$ the element $q \,\&\, [\![a = c]\!] \in Q$. Hence, it must be shown that

$$[\![q \downharpoonleft a = c]\!] = q \,\&\, [\![a = c]\!]$$

for each $c \in A$, in which the equality relation is that of the quantal set $U(A)$ underlying the sheaf A.

In one direction, it must be shown that

$$E(q \downharpoonleft a) \,\&\, p \,\&\, Ec \le q \,\&\, [\![a = c]\!]$$

for any $p \in Q$ for which $(q \downharpoonleft a) \upharpoonleft p = p \downharpoonleft c$ in the sheaf A. Given that

$$q \,\&\, [\![a = c]\!] = q \,\&\, \bigvee_{a|p'=p'|c} Ea \,\&\, p' \,\&\, Ec,$$

it suffices, for any such $p \in Q$, to find $p' \in Q$ satisfying $a \upharpoonleft p' = p' \downharpoonleft c$ in the sheaf A, for which $E(q \downharpoonleft a) \,\&\, p \,\&\, Ec \le q \,\&\, Ea \,\&\, p' \,\&\, Ec$. Taking $p' = Ea \,\&\, q \,\&\, p \in Q$ and observing that $E(q \downharpoonleft a) = q \,\&\, Ea$, we note that $(q \downharpoonleft a) \upharpoonleft p = q \downharpoonleft (a \upharpoonleft p) = q \,\&\, Ea \,\&\, p \downharpoonleft a = q \,\&\, p \,\&\, Ea \downharpoonleft = q \,\&\, p \downharpoonleft a$, which implies that $q \,\&\, p \downharpoonleft a = p \downharpoonleft c$. Hence, on the one hand, one has that

$$a \upharpoonleft Ea \,\&\, q \,\&\, p = Ea \,\&\, q \,\&\, p \downharpoonleft a = Ea \,\&\, q \,\&\, q \,\&\, p \downharpoonleft a = Ea \,\&\, q \,\&\, p \downharpoonleft c.$$

While, on the other, evidently $q \,\&\, Ea \,\&\, p \,\&\, Ec \le q \,\&\, Ea \,\&\, (Ea \,\&\, q \,\&\, p) \,\&\, Ec$; by the right symmetricity of the quantale Q. Hence,

$$[\![q \downharpoonleft a = c]\!] \le q \,\&\, [\![a = c]\!].$$

For the converse, it must be shown that, for any $p \in Q$ for which $a \upharpoonleft p = p \downharpoonleft c$, one has that

$$q \,\&\, Ea \,\&\, p \,\&\, Ec \le [\![q \downharpoonleft a = c]\!].$$

Given that $[\![q \downharpoonleft a = c]\!] = \bigvee_{(q|a)|p'=p'|c} E(q \downharpoonleft a) \,\&\, p' \,\&\, Ec$, it suffices, for any such $p \in Q$, to find $p' \in Q$ satisfying $(q \downharpoonleft a) \upharpoonleft p' = p' \downharpoonleft c$ in the sheaf A, for which $q \,\&\, Ea \,\&\, p \,\&\, Ec \le E(q \downharpoonleft a) \,\&\, p' \,\&\, Ec$. Taking $p' = q \,\&\, p \in Q$, we note that indeed, on the one hand, one has that

$$\begin{aligned}(q \downharpoonleft a) \upharpoonleft q \,\&\, p &= q \downharpoonleft (a \upharpoonleft q \,\&\, p) = q \downharpoonleft ((a \upharpoonleft q) \upharpoonleft p) = ((q \downharpoonleft a) \upharpoonleft q) \upharpoonleft p \\ &= (q \,\&\, Ea \,\&\, q \downharpoonleft a) \upharpoonleft p = q \downharpoonleft (a \upharpoonleft p) = q \downharpoonleft (p \downharpoonleft c) \\ &= q \,\&\, p \downharpoonleft c.\end{aligned}$$

While, on the other, evidently $q \,\&\, Ea \,\&\, p \,\&\, Ec \le q \,\&\, Ea \,\&\, (q \,\&\, p) \,\&\, Ec$, once again by the right symmetricity of the quantale Q. Hence,

$$q \,\&\, [\![a = c]\!] \le [\![q \downharpoonleft a = c]\!],$$

giving the required equality for any $c \in A$. Hence, the left restriction (and so the right restriction) on the sheaf $\Gamma U(A)$ is exactly that of the sheaf A. The functor ΓU is therefore the identity on objects.

Suppose finally that $f : A \to B$ is a map of sheaves on the quantale Q. Recall that the map

$$U(f) : U(A) \to U(B)$$

of quantal sets that it determines is given by the functional relation $U(f) : A \times B \to Q$ defined by

$$U(f)(a, b) = [\![f(a) = b]\!]$$

for each $(a, b) \in A \times B$. By the remarks made in the preceding section concerning the underlying map of presheaves determined by any map of complete quantal sets, the map

$$\Gamma U(f) : \Gamma U(A) \to \Gamma U(B)$$

then assigns to each $a \in A$ the element $b \in B$ uniquely defined by the property that

$$[\![b = c]\!] = U(f)(a, c)$$

for each $c \in B$. However, by the above remarks, $U(f)(a, c) = [\![f(a) = c]\!]$ for each $c \in B$. Hence, $f(a) \in B$ is indeed the required element of the sheaf B, giving that $\Gamma U(f)$ is exactly f. The functor ΓU is therefore the identity on maps, which completes the proof. ▯

This isomorphism, rather than equivalence, of categories states that sheaves on a quantale Q and complete quantal sets over Q are equivalent mathematical structures, interrelated by such identities as that,

$$a \upharpoonleft [\![a = b]\!] = [\![a = b]\!] \downharpoonleft b$$

for all $a, b \in A$, observed earlier. The significance of this for the category of sheaves on Q will be examined further later. For the moment, we note that the existence of the isomorphism allows us to express the adjointness derived in the preceding section is a more familiar form:

COROLLARY. *For any right Gelfand quantale Q, there are adjoint functors*

$$\textbf{Presheaves}\ /Q \rightleftarrows \textbf{Sheaves}\ /Q,$$

from the category of presheaves on Q to the category of sheaves on Q, of which the coadjoint is the canonical embedding. ▯

The adjoint is, of course, the functor which assigns to each presheaf A on Q the canonical sheaf of the completion of the quantal set underlying the presheaf A. This functor, the *sheafification functor* on the category of presheaves on Q applies the identification of compatible families of elements of a presheaf with singleton subsets of the quantal set that it canonically determines to allow the completion of the presheaf to a sheaf, exactly as in the localic case [6].

6. POSTLUDE

Over the preceding sections, we have described the way in which concepts of quantal set and of presheaf can be defined with respect to a right Gelfand quantale Q. The ideas have been intertwined, both in terms of motivation, and in terms of development of the notions of complete quantal set and of sheaf, in exactly the way that one observes classically in the case of a locale L. The adjoint functors

$$\textbf{Presheaves}\ /Q \rightleftarrows \textbf{Sheaves}\ /Q \text{ and } Q\textbf{-Sets} \rightleftarrows \textbf{Complete}\ Q\textbf{-Sets}$$

have been arrived at not in isolation, but interlinked through what has become the isomorphism

$$\textbf{Sheaves}\ /Q \rightleftarrows \textbf{Complete}\ Q\textbf{-Sets}$$

between the categories of sheaves on Q, and of complete quantal sets over Q.

One aspect of this that has, as yet, received little mention in its own right is that of the concept of a subset of a quantal set over Q. Through the idea of a singleton subset, it has played a critical role in developing the theory, leading explicitly to the concept of completeness of a quantal set, and, through that, motivating the definition of a sheaf on the quantale Q. Equally importantly, it has shown the quantale itself to be recoverable from its quantal sets, since the subsets of the complete quantal set $\mathbb{1}_Q$ to the elements of the quantale Q, just as in the case of a locale.

What has been left unsaid about the concept of a subset of a quantal set is that it does not coincide with that of subobject in the category of quantal sets. It is at this point that we depart from that with which one is familiar in the case of local sets. Althought this will be explored in a later paper, it is perhaps important to note the extent to which these changes one's perceptions of the categories involved, since these changes are extensive. An interesting case to bear in mind in what follows is that of the right Gelfand quantale obtained from the spectrum

$$\text{Max}\ \ M_n(\mathbb{C})$$

of a finite-dimensional C*-algebra.

The subsets of a quantal set A have a natural structure as a quantale, and with this comes the notion of a subset $s : A \to Q$ being *two-sided* in the event that

$$s(a) \in I(Q)$$

for each $a \in A$. It may also be remarked that the subsets of A also have a natural structure as a quantal set over Q, in terms of which two-sidedness may also be defined. The subobjects of the quantal set A in the category of quantal sets over Q correspond bijectively with the two-sided subsets of A, with each subset having a two-sided closure that defines, and is defined by, a subobject in the category.

Having in mind the quantale already mentioned, we may note that in that case, the right side

$$R(\text{Max}\ \ M_n(\mathbb{C}))$$

is the quantale whose elements correspond to the linear subspaces of $\mathbb{C}^n$. The locale

$$I(\text{Max}\ \ M_n(\mathbb{C})),$$

however, has exactly two elements, corresponding to the closed ideals of $M_n(\mathbb{C})$, namely the zero ideal, and the matrix ring itself. In particular, the quantal set

$$\mathbb{1}_{\text{Max}\ M_n(\mathbb{C})}$$

has subsets corresponding bijectively with the linear subspaces of $M_n(\mathbb{C})$, but exactly two subobjects in the category of quantal sets, namely the empty subset and the quantal set itself.

The concept of subset of a quantal set A over a quantale Q is not therefore intrinsic to the category of quantal sets over Q. It may be defined externally, which is the sense in which it has been dealt with in this paper. It may, however, equally be considered to be internal to the category, provided that the category is endowed with structure in the form of a subset classifier determined by the quantale Q. The details of this, directly generalising the situation in the case of a locale, will be found in a later paper [14]. The existence of this structure on the category of quantal sets, or equivalently the category of sheaves on the quantale Q, is fortuitous, in the face of another realisation concerning this category.

It will have been evident, throughout, that extent, whether of an element of a quantal set, or of an element of a sheaf, plays an important role in allowing, by the right symmetricity of the quantale, all manner of interchange to take place to its right. At the most basic level, it is its presence, on the left of expressions in certain of the axioms, that allows the composite of functional relations between quantal sets to be proved again functional. In terms of sheaves, the extent enters naturally in the condition that describes the extent of a restriction: namely, that

$$E(p \downharpoonleft a) = p \downharpoonleft Ea.$$

This axiom was not arbitrarily chosen, however reasonable it might be: it is present in the motivating instance of the completion of any quantal set, waiting to be noted down and axiomatised.

When working with quantal sets over a locale, or sheaves on a locale, conditions of this kind are considered to be part of an accounting process that keeps track of extent while restricting an element. Once the quantale concerned is not a locale, the condition has a more sinister, in the technical sense, contribution to make. Consider, for instance, the act of taking an element $a \in A$ of a sheaf on the quantale Q, and restricting it on the left by the identity element $1_Q \in Q$ of the quantale. One pauses to remember, of course, that this possibility of restricting an element of a sheaf to *any* element of the quantale, rather than those lying below its extent, is inherited from the logical, rather than the geometrical, way of looking at sheaves.

Now, look at the effect of this restriction on the extent: the element $a \in A$ of extent $Ea \in Q$ under restriction becomes the element $1_Q \downharpoonleft a \in A$ with extent

$$E(1_Q \downharpoonleft a) = 1_Q \& Ea \geq Ea$$

which is the two-sided closure of $Ea \in Q$, in the sense of the smallest two-sided element of the quantale containing the element $Ea \in Q$. Observe that by *restricting* the element $a \in A$, we have *extended* its extent. In the case of the quantale

$$R(\text{Max } M_n(\mathbb{C})),$$

an element whose extent corresponds to a one-dimensional subspace of $\mathbb{C}^n$ is extended to an element whose extent corresponds to $\mathbb{C}^n$ itself.

The conclusion to which this leads is expressed in the following:

THEOREM. *For any right Gelfand quantale Q, the category of sheaves on Q is equivalent to the category of sheaves on the locale* $I(Q)$ *of two-sided elements of Q, by the functors*

$$\textbf{Sheaves}/Q \rightleftarrows \textbf{Sheaves}/I(Q)$$

that assign to each sheaf on Q the sheaf on $I(Q)$ *consisting of those elements having two-sided extent, and to any sheaf on* $I(Q)$ *its canonical extension to the quantale Q.*

The proof will be omitted, being relatively straightforward once one has realised this state of affairs. One literally takes, for any sheaf A on the quantale Q, the set of elements with extent in $I(Q)$, together with its natural restrictions over $I(Q)$. Conversely, given any sheaf A over the locale $I(Q)$, one associates with it the sheaf of which elements are those pairs $(q, a) \in Q \times A$ for which $1_Q \& q = Ea \in I(Q)$, with the natural definitions of extent and restrictions. The equivalence established by these functors then follows from the observation that left restriction by the identity of Q provides for each $q \in Q$ an invertible mapping from the set $A(q)$ of elements of A of extent $q \in Q$ and the set $A(1_Q \& q)$ of elements of extent $1_Q \& q \in I(Q)$

. ▯

The case of the quantale

$$R(\text{Max } M_n(\mathbb{C}))$$

again provides an insight into what is happening. The category of sheaves on the quantale is now seen to be equivalent to the category of sheaves on the locale $I(\text{Max } M_n(\mathbb{C}))$, which is the locale $\mathbb{1}$ corresponding to the topological space with a single point. The category of sheaves on the quantale is therefore equivalent to the category of sets. It is, however, the category of sets together with the structure afforded by the subset classifier determined by the right Gelfand quantale $R(\text{Max } M_n(\mathbb{C}))$. In particular, we see that the category of sets may be endowed with a multiplicity of independent structures of this kind, from each of which the quantale concerned may be recovered by taking the quantale of subsets, in the sense determined by that structure, of the one-element set.

More generally, we have the following:

COROLLARY. *For any right Gelfand quantale Q, the category of sheaves*

$$\textbf{Sheaves}/Q$$

on the quantale Q is a topos.

One remarks that the quantale $I(Q)$ is actually a locale, from which the result follows. ▯

The observation that the category

$$\textbf{Sheaves}/Q$$

is a topos is, however, one that is, in some sense, inappropriate, in that it introduces only one aspect of the structure that is canonically present. For the category of sheaves on Q also admits a subset classifier, distinct from the subobject classifier which the category happens to have through being a topos. The subset classifier characterises the characteristic maps of subsets, while the subobject classifier performs the same function for two-sided subsets. It is the subset classifier which is the primary structure of interest in the category, carrying with it the logical structure of the category of sheaves on Q. It is to this that we return in the sequel [14] to this paper.

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