On the Quantisation of Spaces

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To Max Kelly, with fond respect, on his 70th birthday

1. INTRODUCTION

This paper is concerned with applying the concept of point introduced in [16] to establishing a concept of space within the context of involutive unital quantales. The aim in so doing is to show that the concept of locale can be extended successfully into the non-commutative context of the quantum world [11] . In particular, we wish to show that the concept of quantale provides an adequate and elegant framework [12] for developing the insights of Giles and Kummer [9] and of Akemann [1] into the spectral theory of C*-algebras. Within the context of quantales, it is hoped that in turn this gives some insight into the way in which the concepts of point and of space may be developed in situations without an involution being present.

The aim will be to determine conditions under which an involutive unital quantale X should be considered spatial, that is, to provide the quantised topology of a quantal space. The intuitive meaning of this is that it should admit a canonical homomorphism

$$\tau : X \to Q$$

of an appropriate kind into an involutive unital quantale Q that is to be considered as the quantised version of the power set of the underlying set of the space. In particular, this involves seeking an appropriately quantised version of the concept of a complete atomic Boolean algebra.

The insight of Giles and Kummer [9] was that the weakly dense embedding

$$\psi : A \to B$$

of a C*-algebra A into the atomic von Neumann algebra B obtained by taking the product of its irreducible representations on Hilbert space provided the algebraic counterpart of such a non-commutative space. Indeed, observing that the irreducible representations of A on Hilbert space may be taken to be those of the form

$$\psi_m : A \to \mathscr{B}(A/m)$$

indexed by the maximal right ideals m of A , this can immediately be seen to be a generalisation of the Gelfand representation of a commutative C*-algebra [8] to the non-commutative case. More particularly, the assignment to each closed right ideal I of the C*-algebra A of the weak closure $\overline{\psi(I)}^{w}$ of its image in the atomic von Neumann algebra B was considered to describe a non-commutative topology, in terms of which could be established a Gelfand representation of the C*-algebra A , precisely generalising that of the commutative case.

The introduction of the concept of quantale [11] allowed these observations to be considered within a categorical context naturally extending that of the locales within which Gelfand duality may properly be viewed constructively in the commutative case [2,3,4] . With any C*-algebra A may be associated its spectrum

$$\mathrm{Max}\, A \ ,$$

obtained [12] by taking the quantale of closed linear subspaces of A . The irreducible representations of the C*-algebra A determine the points of the spectrum $\mathrm{Max}\, A$ in the sense introduced in [16] . With the atomic von Neumann algebra B , determined by the irreducible representations of A on Hilbert space, may be associated its weak spectrum

$$\mathrm{Max}_w\, B \ ,$$

obtained [19] by taking the quantale of weakly closed linear subspaces of B . The weakly dense embedding of the C*-algebra A in the atomic von Neumann algebra B yields a homomorphism

$$\mathrm{Max}\, \psi_w : \mathrm{Max}\, A \to \mathrm{Max}_w\, B$$

by assigning to each closed linear subspace of A the weak closure of its image in B .

The observation that the canonical embedding

$$\psi : A \to B$$

of the C*-algebra A into the atomic von Neumann algebra B is weakly dense means that the homomorphism

$$\mathrm{Max}\, \psi_w : \mathrm{Max}\, A \to \mathrm{Max}_w\, B$$

restricts to the right sides of the quantales concerned, yielding a canonical embedding

$$\mathrm{R}\,(\mathrm{Max}\, \psi_w) : \mathrm{R}\,(\mathrm{Max}\, A) \to \mathrm{R}\,(\mathrm{Max}_w\, B)$$

that is exactly that of the non-commutative topology identified by Giles and Kummer. In this sense, the homomorphism

$$\mathrm{Max}\, \psi_w : \mathrm{Max}\, A \to \mathrm{Max}_w\, B$$

of quantales represents the Gelfand topology of the spectrum $\mathrm{Max}\, A$ of the C*-algebra A in terms of the discrete topology represented by the weak spectrum $\mathrm{Max}_w\, B$ of the atomic von Neumann algebra B .

Motivated by these observations, the aim will be to describe those involutive unital quantales Q that play the rôle of a discrete quantal topology, and to identify the conditions that allow a homomorphism

$$\tau_X : X \to Q_X$$

from an involutive unital quantale X into such a discrete quantale Q_X to define a quantal space. It will be shown that a necessary and sufficient condition for an involutive unital quantale X to admit such a homomorphism, in other words to be spatial, is that, in the appropriate sense, it has enough points. Moreover, it will turn out in this case that X is necessarily a Gelfand quantale.

In the case of locales, considered as involutive unital quantales by taking the identity involution, these concepts just reduce to those known classically. The locales that yield discrete quantal spaces will be exactly the complete atomic Boolean algebras. Those that are spatial will be exactly those yielding the topology of a classical topological space. Moreover, as was shown in our earlier paper [16] , the points of a locale considered as an involutive unital quantale are exactly its points in the classical sense. As a consequence of these observations, any quantal space will be seen to have an underlying classical space. More importantly, it will be shown that the spectrum

$$\mathrm{Max}\, A$$

of any C*-algebra A is indeed a quantal space, in other words that the quantale $\mathrm{Max}\, A$ is spatial. In the context of the present volume, nothing could be more appropriate ...

Finally, it should be remarked that although a number of other papers [5,6,7,10,17,18,20] have considered concepts of spatiality, the approach that will be taken here differs fundamentally from those that have come before, basing itself instead on an analysis of the concept of point [15,16] , together with a conviction, inherited from Giles and Kummer [9] , that the spectrum

$$\mathrm{Max}\, A$$

of a C*-algebra A is an instructive instance of the concept of quantal space. In particular, it should be noted that the concept of quantale considered in [5,6,7,20] is not that originally proposed [11] , and that usage of the term discrete quantale in [18] and in [20] differ not only from each other but from that applied here. Nevertheless, it must be said that the approach taken by Rosický in [20] , albeit in a differently axiomatised and motivated context, is closest to that considered here. Having said that, it is notable that in subsequent papers [10,17,18] these insights appeared to be forsaken in favour of approaches to spatiality that seem intrinsically less promising.

2. Gelfand Quantales

The quantales with which this paper will be concerned are those satisfying the following:

Definition. A quantale Q will be said to be *involutive* provided that there is given an involution $*$ satisfying the conditions that

$$a^{**} = a \ ,$$

$$(a \& b)^* = b^* \& a^* \ \text{, and}$$

$$(\textstyle\bigvee_i a_i)^* = \bigvee_i a_i^*$$

for any $a, b \in Q$ and any $a_i \in Q$, and *unital* provided that there is given an element $e_Q \in Q$ satisfying the condition that

$$e_Q \& a = a = a \& e_Q$$

for any $a \in Q$.

It may be recalled that an element $a \in Q$ of an involutive quantale Q is said to be *self-adjoint* provided that

$$a = a^* \ ,$$

and a *projection* provided that it is also *idempotent*, in the sense that

$$a \& a = a \ .$$

The *unit* $e_Q \in Q$ of an involutive unital quantale Q is always self-adjoint and idempotent, hence a projection.

An element $a \in Q$ of a quantale Q is said to be *right-sided* provided that

$$a \& 1_Q \leq a \ ,$$

in which $1_Q \in Q$ denotes the top element of the sup-lattice Q . In the case of a unital quantale Q , this condition necessarily implies the equality of these elements. The subset of Q consisting of right-sided elements will be denoted by

$$\mathrm{R}(Q) \ ,$$

while that of left-sided elements similarly defined will be denoted by $\mathrm{L}(Q)$. Those elements that are both left-sided and right-sided will be said to be *two-sided*, the subset of two-sided elements of Q being denoted by

$$\mathrm{I}(Q) \ .$$

It may be noted that each of these subsets is closed under arbitrary joins in the quantale Q . Moreover, that in an involutive quantale Q , the involution maps left-sided elements to right-sided elements, and *vice versa*. Indeed, the involution in this way determines an isomorphism of sup-lattices between $\mathrm{R}(Q)$ and $\mathrm{L}(Q)$. The sup-lattice $\mathrm{I}(Q)$ of two-sided elements is closed under the involution, hence is said to be *self-adjoint*.

It is evident that any locale L is an involutive unital quantale, when taken together with the involution given by the identity mapping on L . In this case, the unit element $e_L \in L$ coincides with the top element $1_L \in L$, and every element of the locale is both self-adjoint and idempotent, hence a projection. Moreover, every element of the locale L is both left-sided and right-sided, hence the subsets $\mathrm{R}(L)$, $\mathrm{L}(L)$, and $\mathrm{I}(L)$ each coincide with the locale L .

The involutive unital quantale that we shall throughout have particularly in mind is recalled ([12,11]) in the following:

DEFINITION. By the *spectrum* $\operatorname{Max} A$ of a C*-algebra A is meant the quantale of closed linear subspaces of A, together with the operations of product and join given by taking

$$M \& N = \overline{M.N}\ \text{, and}$$

$$\bigvee_i M_i = \overline{\sum_i M_i}$$

for any $M, N \in \operatorname{Max} A$ and $M_i \in \operatorname{Max} A$. The spectrum is involutive with respect to the involution that assigns to each closed linear subspace M its elementwise involute

$$M^* = \{\, a^* \in A \mid a \in M \,\}\ ,$$

and unital, with unit given by the closed linear subspace

$$e_{\operatorname{Max} A} = \langle 1_A \rangle$$

generated by the unit $1_A \in A$ of the C*-algebra A.

It may be noted that in the case of the spectrum

$$\operatorname{Max} A$$

of a C*-algebra A, the right-sided elements of the quantale are exactly the closed right ideals of A, with similar remarks for the left-sided and two-sided elements, since the top element of the quantale is exactly the closed linear subspace given by A itself. The sup-lattice

$$\mathrm{R}(\operatorname{Max} A)$$

of right-sided elements of the spectrum is therefore exactly that considered by Giles and Kummer [9] as consisting of the open sets of a non-commutative spectral topology determined by the C*-algebra A.

As is the case with locales, the category with which one is concerned depends on whether one is working with quantales algebraically or geometrically. Although in this paper the motivation is topological, the approach taken is almost entirely algebraic, leading us to the following:

DEFINITION. By a *homomorphism* of involutive quantales is meant a homomorphism

$$\varphi : Q \to Q'$$

of quantales that preserves the involution in the sense that

$$\varphi(a^*) = \varphi(a)^*$$

for each $a \in Q$. The homomorphism is said to be *unital* provided that it also satisfies

$$e_{Q'} \leq \varphi(e_Q)\ .$$

The decision to require a homomorphism

$$\varphi : Q \to Q'$$

of involutive unital quantales only to map the unit of Q above the unit of Q' is taken for a number of reasons, partly logical and partly mathematical, with which we need not be concerned at this point. The interested reader is referred to [16] and [11] for further information. In many cases, the homomorphisms with which we shall be concerned turn out to preserve the unit strictly. The position is simply that, one the one hand we need not assume this, while on the other there are situations when the homomorphisms that we need to consider fail to satisfy the stricter condition of preserving the unit.

Of course, the homomorphisms

$$\varphi : L \to L'$$

of locales considered as involutive unital quantales are exactly the homomorphisms of the corresponding locales. Indeed, in this case, the unit is preserved strictly. Again, the homomorphism

$$\operatorname{Max} \varphi : \operatorname{Max} A \to \operatorname{Max} A'$$

of involutive unital quantales from the spectrum $\operatorname{Max} A$ to the spectrum $\operatorname{Max} A'$ defined for any homomorphism

$$\varphi : A \to A'$$

of C*-algebras, by assigning to each closed linear subspace M of the C*-algebra A the closure

$$(\operatorname{Max} \varphi)(M) = \overline{\varphi(M)}$$

in the C*-algebra A' of its image under the homomorphism of C*-algebras, is strictly unital.

Any involutive unital quantale Q admits a *right pseudo-orthocomplement* defined ([13,15]) on the sup-lattice $\mathrm{R}(Q)$ of right-sided elements of Q by writing

$$a^{\perp} = \bigvee_{a^* \& b = 0_Q} b$$

for each $a \in \mathrm{R}(Q)$. Observe that the elements $b \in Q$ over which the join is taken may be constrained to be right-sided, since $a^* \& b = 0_Q$ implies that $a^* \& b \& 1_Q = 0_Q$, and clearly $b \leq b \& 1_Q$. The assignment to each $a \in \mathrm{R}(Q)$ of the element $a^{\perp} \in \mathrm{R}(Q)$ may straightforwardly be shown to satisfy the conditions

$$\begin{aligned} a &\leq a^{\perp\perp} \\ (\bigvee a_i)^{\perp} &= \bigwedge a_i^{\perp} \\ a \wedge a^{\perp} &= 0_Q \end{aligned}$$

defining a pseudo-orthocomplement on the sup-lattice $\mathrm{R}(Q)$. Similarly, a *left pseudo-orthocomplement* may be defined on the sup-lattice $\mathrm{L}(Q)$ of left-sided elements of Q by writing

$$^{\perp}a = \bigvee_{b \& a^* = 0_Q} b$$

for each $a \in \mathrm{L}(Q)$, in which once again the elements over which the join is taken may be constrained to be left-sided. It is straightforward to show that the involution of the involutive unital quantale converts right pseudo-orthocomplementation into left pseudo-orthocomplementation, and *vice versa*. It may be remarked that in the case of a locale L these definitions each coincide with the pseudo-complement of the locale.

Amongst the involutive unital quantales, those with which we shall be particularly concerned are those given ([13]) by the following:

Definition. An involutive unital quantale Q will be said to be a *Gelfand quantale* provided that the condition

$$a = a \& a^* \& a$$

is satisfied by each right-sided element $a \in \mathrm{R}(Q)$.

Observe firstly that the concept of Gelfand quantale may equally be stated in terms of left-sided elements of the involutive unital quantale Q , by the symmetry between right and left sides of the quantale under involution. Evidently, the condition that defines a Gelfand may equally be expressed in a way which is entirely equational, yielding that Gelfand quantales are categorically well-behaved amongst involutive unital quantales. The condition may also be remarked [13] to be equivalent to the requirement that each right-sided element $a \in \mathrm{R}(Q)$ is that generated by a projection $p \in Q$, in the sense that

$$a = p \& 1_Q ,$$

which may again be seen to be equivalent to that stated in terms of left-sided elements. In this sense, the condition defining a Gelfand quantale may be seen as yielding an overall coherence in the structure of the quantale.

Any locale L is necessarily a Gelfand quantale, by the observation that the product is just the meet, and that the involution is trivial. Moreover, it may be shown that a Gelfand quantale Q is a locale precisely in the case that the unit element

$$e_Q \in Q$$

is actually the top element $1_Q \in Q$. More generally, for any Gelfand quantale Q , any right-sided element $a \in \mathrm{R}(Q)$ (and similarly any left-sided element $a \in \mathrm{L}(Q)$) is necessarily idempotent. Moreover, any two-sided element $a \in \mathrm{I}(Q)$ is self-adjoint, hence a projection in the involutive quantale Q . Indeed, the quantale

$$\mathrm{I}(Q)$$

obtained by restricting the product $\&$ and the join $\bigvee$ of Q to its two-sided elements is actually a locale [13] . It may be noted that in any Gelfand quantale the right and left pseudo-orthocomplements each coincide on the locale $\mathrm{I}(Q)$ to yield its pseudo-complement.

Finally, for any C*-algebra A , the spectrum $\mathrm{Max}\, A$ may be seen ([12,11]) to be a Gelfand quantale, by the existence of an approximate unit in any closed right ideal of the C*-algebra A . In the case of a commutative C*-algebra A , the spectrum in the classical sense is exactly given by the locale

$$\mathrm{I}(\mathrm{Max}\, A)$$

of the Gelfand quantale $\mathrm{Max}\, A$.

3. Von Neumann Quantales.

The Gelfand quantales just introduced will later be seen to be particularly closely linked to the concept of a quantal space. Already, the extent to which they extend the concept of locale may be seen. At this point, we need to apply this observation to identify amongst the Gelfand quantales those which appear to play the rôle of the power set of a quantal set, in the following:

Definition. By a *von Neumann quantale* Q is meant a Gelfand quantale satisfying the condition that

$$a^{\perp\perp} = a$$

for each $a \in \mathrm{R}(Q)$.

Observe firstly that, as in the case of Gelfand quantales, the concept of von Neumann quantale may equally well be stated in terms of the left pseudo-orthocomplement of Q . In the case of a von Neumann quantale Q , the right pseudo-orthocomplement is actually an *orthocomplement* on the sup-lattice $\mathrm{R}(Q)$ in the sense that

$$\begin{aligned} a^{\perp\perp} &= a \\ (\bigvee a_i)^{\perp} &= \bigwedge a_i^{\perp} \\ a \vee a^{\perp} &= 1_{\mathrm{R}(Q)} \\ a \wedge a^{\perp} &= 0_{\mathrm{R}(Q)} , \end{aligned}$$

for any $a \in \mathrm{R}(Q)$ and $a_i \in \mathrm{R}(Q)$. Indeed, a von Neumann quantale Q is exactly a Gelfand quantale for which the subset $\mathrm{R}(Q)$ of right-sided elements is actually an orthocomplemented sup-lattice with respect to the orthocomplement defined by

$$a^{\perp} = \bigvee_{a^* \& b = 0_Q} b$$

for each $a \in \mathrm{R}(Q)$. It may be remarked that a locale L is a von Neumann quantale exactly if it is a complete Boolean algebra. More generally, one has the following well-known result:

Proposition 3.1. *For any von Neumann quantale Q , the locale $\mathrm{I}(Q)$ of two-sided elements of Q is a complete Boolean algebra.*

Proof. It suffices to remark only that the pseudo-complement on the locale $\mathrm{I}(Q)$ is exactly the restriction of the pseudo-orthocomplement on $\mathrm{R}(Q)$. For, given any $a \in \mathrm{I}(Q)$, one has that

$$a^{\perp} = \bigvee_{a^* \& b = 0_Q} b ,$$

where now the join may be taken over $b \in \mathrm{R}(Q)$ that are actually two-sided. Observing that $a \in \mathrm{I}(Q)$ implies that $a = a^*$, one is therefore taking the join over all $b \in \mathrm{I}(Q)$ for which $a \& b = 0_Q$. But since the product of the quantale Q restricts on the locale $\mathrm{I}(Q)$ to its meet, this yields just the pseudo-complement

$$\neg a = \bigvee_{a \wedge b = 0_{\mathrm{I}(Q)}} b$$

of the locale $\mathrm{I}(Q)$. Since $a^{\perp\perp} = a$ in the von Neumann quantale Q, it follows that $\neg\neg a = a$ in the locale $\mathrm{I}(Q)$, which is therefore a complete Boolean algebra. □

Although for the sake of completeness we have proved this directly, it follows of course from the observation above that the lattice $\mathrm{I}(Q)$ of two-sided elements of a Gelfand quantale Q is a locale [16] , or indeed from the observations of [18] . Its importance here is in identifying the concept of von Neumann quantale as a non-commutative generalisation of that of a complete Boolean algebra.

As a first example of a von Neumann quantale, we note ([19]) the following:

Definition. By the *weak spectrum* $\mathrm{Max}_w\, B$ of a von Neumann algebra B is meant the quantale of weakly closed linear subspaces of B, together with the operations of product and join given by

$$M \& N = \overline{M.N}^{w}, \text{ and}$$
$$\bigvee_i M_i = \overline{\textstyle\sum_i M_i}^{w}$$

for any $M, N \in \mathrm{Max}_w\, B$ and $M_i \in \mathrm{Max}_w\, B$.

Again, the weak spectrum is involutive with respect to elementwise involution of a weakly closed linear subspace of B, and unital with respect to the unit given by the weakly closed subspace generated by the unit element of the von Neumann algebra B. Since the right-sided elements of the weak spectrum $\mathrm{Max}_w\, B$ of a von Neumann algebra B are exactly the weakly closed right ideals of B, it is again the case that

$$\mathrm{Max}_w\, B$$

is a Gelfand quantale, in this case since each weakly closed right ideal is generated by a unique projection of the von Neumann algebra B. Indeed, the sup-lattice

$$\mathrm{R}\,(\mathrm{Max}_w\, B)$$

of weakly closed right ideals of B may be identified with the sup-lattice of projections of the von Neumann algebra B. In view of this, the sup-lattice of right-sided elements of $\mathrm{Max}_w\, B$ may be seen naturally to carry an orthocomplement induced by that on the projections of B.

Another example of a von Neumann quantale, indeed one that will be seen to be intrinsic to any consideration of Gelfand quantales, is given by recalling ([15]) the following:

Definition. By the *Hilbert quantale* $\mathcal{Q}(S)$ of an orthocomplemented sup-lattice S is meant the quantale of sup-preserving mappings

$$\alpha : S \to S\,,$$

together with the product and join defined by

$$s\,(\alpha \& \beta) = (s\,\alpha)\beta\ , \text{ and}$$
$$s\,(\textstyle\bigvee_i \alpha_i) = \bigvee_i s\,\alpha_i$$

for each $s \in S$, and for each $\alpha, \beta \in \mathcal{Q}(S)$ and $\alpha_i \in \mathcal{Q}(S)$.

The Hilbert quantale $\mathcal{Q}(S)$ of an orthocomplemented sup-lattice S is evidently a unital quantale, with unit given by the identity mapping on S. Moreover, any Hilbert quantale

$$\mathcal{Q}(S)$$

is an involutive unital quantale with respect to the involution defined for each $\alpha \in \mathcal{Q}(S)$ by writing:

$$s\,\alpha^{*} = \left(\bigvee_{t\,\alpha \leq s^{\perp}} t\right)^{\perp}$$

for each $s \in S$.

It may be verified by straightforward calculation [15] that the right- and the left-sided elements of the Hilbert quantale $\mathcal{Q}(S)$ of an orthocomplemented sup-lattice S are respectively those defined for each $t \in S$ by:

$$s\,\lambda_t = \begin{cases} 1_S & \text{unless} \\ 0_S & s \leq t \end{cases}$$

and

$$s\,\kappa_t = \begin{cases} t & \text{unless} \\ 0_S & s = 0_S \end{cases}\quad ,$$

for each $s \in S$. Indeed, the assignments to each $t \in S$ of the corresponding right- and left-sided elements of $\mathcal{Q}(S)$ yield canonical isomorphisms

$$\lambda_S : S^{op} \to \mathrm{R}(\mathcal{Q}(S))$$
$$\text{and} \quad \kappa_S : S \to \mathrm{L}(\mathcal{Q}(S))$$

of sup-lattices. Moreover, it may be verified directly that $\mathcal{Q}(S)$ is indeed a Gelfand quantale, and, since the canonical isomorphisms may be seen to preserve orthocomplementation, a von Neumann quantale [15] .

For any von Neumann quantale Q, one may consider the Hilbert quantale

$$\mathcal{Q}(\mathrm{R}\,(Q))$$

determined by its orthocomplemented sup-lattice of right-sided elements. There is a canonical homomorphism

$$\mu_Q : Q \to \mathcal{Q}(\mathrm{R}\,(Q))$$

of von Neumann quantales, obtained by assigning to each $a \in Q$ the sup-preserving mapping $\mu_Q(a) : \mathrm{R}(Q) \to \mathrm{R}(Q)$ defined by

$$b\,\mu_Q(a) = a^* \& b$$

for each $b \in \mathrm{R}(Q)$.

Although by the above remarks the orthocomplemented sup-lattice

$$\mathrm{R}(\mathcal{Q}(\mathrm{R}(Q))$$

of right-sided elements of the Hilbert quantale $\mathcal{Q}(\mathrm{R}(Q))$ is canonically isomorphic to the orthocomplemented sup-lattice $\mathrm{R}(Q)$, this isomorphism is not in general obtained by restricting the canonical homomorphism

$$\mu_Q : Q \to \mathcal{Q}(\mathrm{R}(Q))$$

to the right-sided elements of the von Neumann quantale Q. Indeed, it is not generally even the case that it maps right-sided elements of Q to right-sided elements of the Hilbert quantale $\mathcal{Q}(\mathrm{R}(Q))$, although the conditions under which this is the case will shortly be determined.

Abstracting these discussions, we note the following:

Definition. By a *Hilbert quantale* Q will be meant a von Neumann quantale Q which is isomorphic to the Hilbert quantale

$$\mathcal{Q}(S)$$

of any orthocomplemented sup-lattice S.

It may be shown that one has that a von Neumann quantale Q is a Hilbert quantale exactly if the canonical homomorphism

$$\mu_Q : Q \to \mathcal{Q}(\mathrm{R}(Q))$$

is an isomorphism of von Neumann quantales [15] . The property of being a Hilbert quantale is therefore intrinsic to the von Neumann quantale concerned, rather than dependent on some particular isomorphism.

As a first step towards considering the relationship between a von Neumann quantale Q and the Hilbert quantale $\mathcal{Q}(\mathrm{R}(Q))$ that it determines, we note the following:

Corollary 3.2. *For any Hilbert quantale* Q *, the locale* $\mathrm{I}(Q)$ *of two-sided elements of* Q *is isomorphic to the complete Boolean algebra* $\mathbb{2}$ *.*

Proof. Consider the Hilbert quantale $\mathcal{Q}(S)$ of any orthocomplemented sup-lattice S. Observe that any right-sided element $\lambda_t \in \mathcal{Q}(S)$ takes only the two values $0_S \in S$ and $1_S \in S$, while any left-sided element $\kappa_t \in \mathcal{Q}(S)$ takes only the two values $0_S \in S$ and $t \in S$. So, any two sided element of $\mathcal{Q}(S)$ is of the form κ_t for either $t = 0_S$ or $t = 1_S$. Hence, the complete Boolean algebra

$$\mathrm{I}(\mathcal{Q}(S))$$

of two-sided elements consists exactly of the zero element $0_{\mathcal{Q}(S)}$, which maps every element of S to the bottom element of S, and the top element $1_{\mathcal{Q}(S)}$, which maps every element other than the bottom element to the top element of S. Hence, the complete Boolean algebra $\mathrm{I}(\mathcal{Q}(S))$ is isomorphic to the Boolean algebra $\mathbb{2}$. Since the Hilbert quantale Q is isomorphic to the Hilbert quantale $\mathcal{Q}(S)$ for some orthocomplemented sup-lattice S, one has that the locale $\mathrm{I}(Q)$ is necessarily isomorphic to the complete Boolean algebra $\mathbb{2}$. □

In consequence, a necessary condition for the canonical homomorphism

$$\mu_Q : Q \to \mathcal{Q}(\mathrm{R}(Q))$$

to restrict to the canonical isomorphism of orthocomplemented sup-lattices from $\mathrm{R}(Q)$ to $\mathrm{R}(\mathcal{Q}(\mathrm{R}(Q)))$ is certainly that the von Neumann quantale Q has its complete Boolean algebra $\mathrm{I}(Q)$ isomorphic to the complete Boolean algebra $\mathbb{2}$.

4. DISCRETE VON NEUMANN QUANTALES

Having recalled the background concerning quantales needed to do so, we may now examine the approach taken by Giles and Kummer [9] in terms of these ideas. For any C*-algebra A, choosing from each equivalence class of irreducible representations an irreducible representation

$$\psi_i : A \to \mathcal{B}(H_i)$$

of A on a Hilbert space H_i, one may consider the homomorphism

$$\psi : A \to \prod_i \mathcal{B}(H_i)$$

of C*-algebras obtain by taking their product in the category of C*-algebras. The product

$$\prod_i \mathcal{B}(H_i) ,$$

which we shall denote by B, is a von Neumann algebra. Moreover, it is an atomic von Neumann algebra, in the sense that its sup-lattice $\mathcal{P}(B)$ of projections is atomic. Since the sup-lattice of projections is isomorphic to that of weakly closed right ideals of B, by the mapping which assigns to each projection $p \in B$ the weakly closed right ideal pB which it generates [21] , it follows that the sup-lattice of weakly closed right ideals of B is also atomic. This, however, is exactly the orthocomplemented sup-lattice

$$\mathrm{R}(\mathrm{Max}_w B)$$

of right-sided elements of the von Neumann quantale given by the weak spectrum $\mathrm{Max}_w B$ of the von Neumann algebra B, motivating the following:

Definition. A von Neumann quantale Q will be said to be *atomic* provided that its orthocomplemented sup-lattice

$$\mathrm{R}(Q)$$

of right-sided elements is atomic.

Concerning atomic von Neumann quantales, we have the following observation:

Proposition 4.1. *For any atomic von Neumann quantale* Q *, the locale*

$$\mathrm{I}(Q)$$

of two-sided elements of Q *is a complete atomic Boolean algebra.*

Proof. It must be shown that every two-sided element $a \in Q$ is a join of atoms of $\mathrm{I}(Q)$. Firstly, we assert that for each atom $x \in \mathrm{R}(Q)$, the element $1_Q \& x \in \mathrm{I}(Q)$ is an atom of $\mathrm{I}(Q)$. Observing that $x \leq 1_Q \& x$ necessarily implies that $1_Q \& x \in \mathrm{I}(Q)$ is non-zero, it remains to show that if $b \in \mathrm{I}(Q)$ is such that $b \leq 1_Q \& x$, then $b = 0_Q$ or $b = 1_Q \& x$. So, observe that $b \leq 1_Q \& x$ implies $x \& b \leq x \& 1_Q \& x = x$ (since $x \in \mathrm{R}(Q)$, hence is idempotent). But $x \& b \in \mathrm{R}(Q)$ then implies that $x \& b = 0_Q$ or $x \& b = x$, since $x \in \mathrm{R}(Q)$ is an atom. In the case that $x \& b = 0_Q$, then $b \leq 1_Q \& x$ implies that $b = b \& b \leq 1_Q \& x \& b = 1_Q \& 0_Q = 0_Q$ by the idempotency of $b \in \mathrm{I}(Q)$, giving $b = 0_Q$. In the case that $x \& b = x$, then $b = 1_Q \& b \geq 1_Q \& x \& b = 1_Q \& x$, which together with $1_Q \& x \geq b$ yields $b = 1_Q \& x$. Hence, $b \leq 1_Q \& x$ implies $b = 0_Q$ or $b = 1_Q \& x$. Thus, $1_Q \& x \in \mathrm{I}(Q)$ is an atom of $\mathrm{I}(Q)$. Now, given any $a \in \mathrm{I}(Q)$, we have that $a = \bigvee_{x \leq a} x$ taken over atoms $x \in \mathrm{R}(Q)$, since the sup-lattice $\mathrm{R}(Q)$ is atomic. Hence, $a = 1_Q \& a = \bigvee_{x \leq a} 1_Q \& x$, in which each $1_Q \& x \in \mathrm{I}(Q)$ is an atom of $\mathrm{I}(Q)$. Thus, the complete Boolean algebra $\mathrm{I}(Q)$ is indeed atomic, as asserted. □

Of course, an important aspect of any complete atomic Boolean algebra is that it can be decomposed into a product of copies of the complete atomic Boolean algebra

$$\mathbb{2} \ .$$

Moreover, this product may be indexed naturally by the atoms of the complete atomic Boolean algebra, with the projection onto each factor of the product achieved by taking the meet of each element with the atom concerned. This aspect of the construction we now extend to von Neumann quantales, beginning with the following, motivated by the corresponding condition on von Neumann algebras:

Definition. A von Neumann quantale Q is said to be a *factor* provided that its Boolean algebra

$$\mathrm{I}(Q)$$

has exactly two elements, in other words is canonically isomorphic to $\mathbb{2}$.

In order for a von Neumann quantale to be able to be decomposed into a product of von

Neumann factor quantales, in particular the corresponding result for the atomic case, we need to consider first the following:

Definition. A projection $p \in Q$ of a Gelfand quantale Q will be said to be *central* provided that

$$p \& a = a \& p$$

for all $a \in Q$.

Proposition 4.2. *For any central projection* $p \in Q$ *of a Gelfand quantale* Q *,*

$$Q_p = \{\, a \in Q \mid p \& a = a \,\}$$

is again a Gelfand quantale.

Proof. Firstly, observe that, by the idempotency of the projection $p \in Q$, Q_p is exactly the subset of the quantale Q of elements of the form $p \& a \in Q$ for any $a \in Q$. Hence, it is evidently closed under the operations of product, join, and involution of the quantale Q , since respectively $(p \& a) \& (p \& b) = p \& (a \& b)$, $\bigvee_i p \& a_i = p \& \bigvee_i a_i$, and $(p \& a)^* = a^* \& p^* = a^* \& p = p \& a^*$. Hence, Q_p is an involutive subquantale of Q . It is unital, with unit $p = p \& p \in Q_p$, since $(p \& a) \& p = p \& (p \& a) = (p \& p) \& a = p \& a$ for any $a \in Q$. Finally, for any $p \& a \in Q_p$, we have that $p \& a \in \mathrm{R}(Q_p)$ provided that $a \in Q$ can be chosen such that $a \in \mathrm{R}(Q)$. Then, $(p \& a) \& (p \& a)^* \& (p \& a) = p \& p \& p \& a \& a^* \& a = p \& a$ by the Gelfand property of the quantale Q , giving that for the quantale Q_p . Hence, Q_p is indeed a Gelfand quantale. □

As an immediate corollary, we have the following observation concerning the case when the Gelfand quantale considered is actually a von Neumann quantale:

Corollary 4.3. *For any central projection* $p \in Q$ *of a von Neumann quantale* Q *, the quantale* Q_p *is again a von Neumann quantale, atomic if* Q *is atomic.*

Proof. We assert that for any $a \in \mathrm{R}(Q)$ the element $p \& a \in \mathrm{R}(Q_p)$ is such that its pseudo-orthocomplement $(p \& a)^{\perp} \in \mathrm{R}(Q_p)$ calculated in the Gelfand quantale Q_p is exactly the element $p \& a^{\perp} \in Q_p$ obtained from the element $a^{\perp} \in \mathrm{R}(Q)$. For, firstly, observe that $(p \& a)^* \& (p \& a^{\perp}) = p \& a^* \& p \& a^{\perp} = p \& a^* \& a^{\perp} = 0_Q$ since $a^* \& a^{\perp} = 0_Q$. Hence, $p \& a^{\perp} \leq (p \& a)^{\perp}$ in Q_p . Conversely, suppose $p \& b \leq (p \& a)^{\perp}$ in Q_p, so that $0_Q = (p \& a)^* \& (p \& b) = p \& a^* \& p \& b = a^* \& (p \& b)$. Thus, $p \& b \leq a^{\perp}$, hence $p \& b = p \& p \& b \leq p \& a^{\perp}$. So, $(p \& a)^{\perp} \leq p \& a^{\perp}$, giving equality. Hence, the right pseudo-orthocomplement of the Gelfand quantale Q_p is exactly that inherited from the von Neumann quantale Q . Hence, Q_p is again a von Neumann quantale.

Finally, suppose that Q is atomic. We assert that the atoms of $\mathrm{R}(Q_p)$ are exactly those atoms of $\mathrm{R}(Q)$ that lie in the subquantale Q_p . For, given any atom $x \in \mathrm{R}(Q)$, note firstly that $p\&x \in \mathrm{R}(Q_p)$ is either zero or is equal to $x \in \mathrm{R}(Q)$, for $p\&x = x\&p \le x\&1_Q = x$ since $p \in Q$ is central and $x \in \mathrm{R}(Q)$. Since $x \in \mathrm{R}(Q)$ is an atom of $\mathrm{R}(Q)$, it follows that either $p\&x = 0_Q$ or $p\&x = x$. In the latter case, then $p\&x \in \mathrm{R}(Q_p)$ is an atom of $\mathrm{R}(Q_p)$ since if $p\&a \le p\&x = x$, then $p\&a = 0_Q$ or $p\&a = x$. Hence, either $p\&x = 0_Q$ or $x = p\&x \in \mathrm{R}(Q_p)$ is an atom of $\mathrm{R}(Q)$ that lies in $\mathrm{R}(Q_p)$. Lastly, for any $p\&a \in \mathrm{R}(Q_p)$, for which it may therefore be supposed that $a \in \mathrm{R}(Q)$, writing $a = \bigvee_{x \le a} x$ taken over atoms $x \in \mathrm{R}(Q)$, we have that $p\&a = \bigvee_{x \le a} p\&x$ in which we omit those atoms of $\mathrm{R}(Q)$ for which $p\&x = 0_Q$. By the above remarks, this expresses $p\&a \in \mathrm{R}(Q_p)$ as the join of atoms of $\mathrm{R}(Q_p)$, as required. Hence, Q_p is indeed an atomic von Neumann quantale. □

Recalling that any two-sided element $q \in \mathrm{I}(Q)$ of a Gelfand quantale Q is necessarily a projection, observe that for any central projection $p \in Q$ of a Gelfand quantale the element $p\&1_Q \in \mathrm{I}(Q)$ is therefore a two-sided projection of Q. In the event that a projection $q \in \mathrm{I}(Q)$ may be written in the form

$$q = p\&1_Q$$

for some central projection $p \in Q$, we shall say that the central projection $p \in Q$ is *associated* with the projection $q \in \mathrm{I}(Q)$. In the case that $q \in \mathrm{I}(Q)$ is actually an atom of $\mathrm{I}(Q)$, we have the following:

Corollary 4.4. *Suppose that $p \in Q$ is a central projection associated with an atom $q \in \mathrm{I}(Q)$ of the complete Boolean algebra*

$$\mathrm{I}(Q)$$

of two-sided elements of a von Neumann quantale Q. Then the von Neumann quantale Q_p is a von Neumann factor quantale.

Proof. It will be shown that

$$\mathrm{I}(Q_p) = \{0_Q, q\},$$

yielding that Q_p is indeed a factor. Observe firstly that indeed $q \in Q_p$, since $p\&q = p\&p\&1_Q = p\&1_Q = q$. Now, suppose that $a \in Q_p$: then $a = p\&a \le p\&1_Q = q$, showing that

$$1_{Q_p} = q.$$

However, by the remarks made earlier, the elements of $\mathrm{I}(Q_p)$ are exactly the elements of $\mathrm{I}(Q)$ that lie in Q_p. Hence, since $q \in \mathrm{I}(Q)$ is an atom of $\mathrm{I}(Q)$, there are exactly $0_Q \in Q_p$ and $q \in Q_p$, as asserted. Thus, Q_p is a von Neumann factor quantale. □

Recalling that for an atomic von Neumann quantale Q, the complete Boolean algebra $\mathrm{I}(Q)$ of two-sided elements is necessarily atomic, we shall denote by

$$\mathrm{Min}(Q)$$

the set of atoms of $\mathrm{I}(Q)$. Suppose now that the atomic von Neumann quantale Q admits an isomorphism

$$Q \to \prod_i Q_i$$

into a product of, then necessarily atomic, von Neumann factor quantales. Considering the isomorphism

$$\mathrm{I}(Q) \to \prod_i \mathrm{I}(Q_i)$$

thereby induced, observe that this is the canonical decomposition of the complete atomic Boolean algebra $\mathrm{I}(Q)$ into a product of copies of the Boolean algebra $\mathbb{2}$. Hence, the product can be taken to be indexed by the set $\mathrm{Min}(Q)$ of atoms of $\mathrm{I}(Q)$. Assigning to each $q \in \mathrm{Min}(Q)$ the inverse image $p_q \in Q$ of the unit of the atomic von Neumann quantale indexed by each $q \in \mathrm{Min}(Q)$, we obtain a family of central projections of Q indexed by the atoms of $\mathrm{I}(Q)$, yielding that Q satisfies the conditions of the following:

Definition. By a *discrete von Neumann quantale* Q will be meant an atomic von Neumann quantale for which there exists a family

$$(p_q)_{q \in \mathrm{Min}(Q)}$$

of central projections of Q, indexed by the set $\mathrm{Min}(Q)$ of atoms of the complete atomic Boolean algebra $\mathrm{I}(Q)$, satisfying the conditions that:

a) $\bigvee_{q \in \mathrm{Min}(Q)} p_q = e_Q$;

b) $p_q\&1_Q = q$ for each $q \in \mathrm{Min}(Q)$.

The family $(p_q)_{q \in \mathrm{Min}(Q)}$ will be referred to as a *central decomposition* of the unit element $e_Q \in Q$.

Observe in passing that the condition that the central projection $p_q \in Q$ is associated with the atom $q \in \mathrm{Min}(Q)$, namely that

$$p_q\&1_Q = q,$$

is exactly that it is a non-zero central projection contained in the down-segment of $q \in Q$. Observing further that a von Neumann factor quantale is discrete exactly if it is atomic, we have the following:

Proposition 4.5. *Any central decomposition $(p_q)_{q \in \mathrm{Min}(Q)}$ of the unit of a discrete von Neumann quantale Q determines an isomorphism*

$$Q \to \prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$$

from Q to a product of discrete von Neumann factor quantales.

Proof. First, observe that for each $q \in \mathrm{Min}(Q)$ there is a canonical homomorphism

$$\pi_q : Q \to Q_{p_q}$$

defined by $\pi_q(a) = p_q \& a$ for each $a \in Q$. From the definition of the operations of product, join, and involution on the quantale Q_{p_q}, it is evident that these operations are preserved under this mapping. Observing that the unit of the quantale Q_{p_q} is the element $p_q \in Q_{p_q}$, we have that $\pi_q(e_Q) = p_q \& e_Q = p_q = e_{Q_{p_q}}$, so the homomorphism is unital. In passing, we note that the homomorphism is strong, since $\pi_q(1_Q) = p_q \& 1_Q = q = 1_{Q_{p_q}}$, by the observations above.

Now, define a homomorphism

$$Q \to \prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$$

from Q into the product quantale $\prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$, obtained by taking the cartesian product together with the componentwise operations of product, join, and involution, by assigning to each $a \in Q$ the element $(p_q \& a)_{q \in \mathrm{Min}(Q)}$ obtained by localising at each component quantale. By the construction of the product quantale, it suffices to show that the mapping has an inverse, as a mapping, then necessarily a homomorphism of involutive quantales. Define then

$$\prod_{q \in \mathrm{Min}(Q)} Q_{p_q} \to Q$$

by assigning to each element $(a_q)_{q \in \mathrm{Min}(Q)} \in \prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$ the element $\bigvee_{q \in \mathrm{Min}(Q)} a_q \in Q$, noting that each quantale Q_{p_q} is an involutive subquantale of the quantale Q. To verify that the mapping is inverse to the canonical homomorphism, note firstly that for any $a \in Q$, the image under the relevant composite is just $\bigvee_{q \in \mathrm{Min}(Q)} a \& p_q = a \& \bigvee_{q \in \mathrm{Min}(Q)} p_q = a \& e_Q = a$, since $(p_q)_{q \in \mathrm{Min}(Q)}$ is a central decomposition of the unit of Q.

Conversely, given an element $(a_q)_{q \in \mathrm{Min}(Q)} \in \prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$, the image under the relevant composite is $((\bigvee_{q \in \mathrm{Min}(Q)} a_q) \& p_r)_{r \in \mathrm{Min}(Q)}$. However,

$$(\bigvee_{q \in \mathrm{Min}(Q)} a_q) \& p_r = \bigvee_{q \in \mathrm{Min}(Q)} a_q \& p_r = \bigvee_{q \in \mathrm{Min}(Q)} a_q \& p_q \& p_r$$

for each $r \in \mathrm{Min}(Q)$, since $a_q = a_q \& p_q$ for $a_q \in Q_{p_q}$. However, unless $q = r$ one has that $p_q \& p_r \leq q \& r = q \wedge r = 0_Q$, since $q, r \in \mathrm{Min}(Q)$ are atoms of the complete atomic Boolean algebra $\mathrm{I}(Q)$, while if $q = r$ then $a_q \& p_q \& p_r = a_r \& p_r = a_r$. Thus, $(\bigvee_{q \in \mathrm{Min}(Q)} a_q) \& p_r = a_r$ for each $r \in \mathrm{Min}(Q)$, giving that the relevant composite is indeed the identity. Hence the canonical homomorphism is indeed an isomorphism of von Neumann quantales, as asserted. □

Of course, in turn, by the observations above, this yields the following:

COROLLARY 4.6. *An involutive unital quantale Q is a discrete von Neumann quantale if, and only if, it is isomorphic to a product of discrete von Neumann factor quantales.* □

In the case of the involutive unital quantale obtained by assigning a locale L the identity involution, this exactly characterises the complete atomic Boolean algebras as those isomorphic to a product of copies of the Boolean algebra $\mathbb{2}$. The discrete von Neumann quantales may therefore be considered to be the generalisations of the complete atomic Boolean algebras to the context of the quantum world.

5. Quantal Points

In our earlier paper concerning the concept of a point of an involutive unital quantale Q, a case was presented for this being captured by the following:

DEFINITION. By an *algebraically irreducible* representation

$$\varphi : Q \to \mathcal{Q}(S)$$

of an involutive unital quantale Q on an atomic orthocomplemented sup-lattice S is meant a homomorphism of involutive unital quantales from Q to the Hilbert quantale $\mathcal{Q}(S)$ of S, satisfying the condition that

$$S = \{\, x\varphi(a) \in S \mid a \in Q \,\}$$

for any atom $x \in S$.

It may be remarked that, by the atomicity of the orthocomplemented sup-lattice S, the condition is evidently equivalent to requiring that given any atoms $x, y \in S$ there exists an element $a \in Q$ for which

$$x\varphi(a) = y .$$

In other words, the representation is algebraically irreducible provided that each atom $x \in S$ is a *cyclic generator* for the representation. It follows immediately that any algebraically irreducible representation is necessarily *irreducible*, in the sense that

$$s\varphi(a) \leq s \text{ for all } a \in Q \text{ implies that } s = 0_S \text{ or } s = 1_S ,$$

in other words that there are no non-trivial elements invariant under the representation [16] .

In general, however, the concept of algebraic irreducibility is stronger than that of irreducibility. Even when the concepts coincide, the observation that this is the case may be nontrivial. Considering the case of the representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(H)$$

of the spectrum $\mathrm{Max}\, A$ of a C*-algebra A on the Hilbert quantale $\mathcal{Q}(H)$ of the orthocomplemented sup-lattice of closed linear subspaces of a Hilbert space H determined by a representation of A on H, the concept of algebraic irreducibility is exactly that of the algebraic irreducibility of the representation of A on the Hilbert space H, in the sense that each nonzero $x \in H$ is a cyclic generator for the representation. In turn, this states that there are no non-trivial linear subspaces of H that are invariant under the representation. On the other hand, the concept of irreducibility is just that of the topological irreducibility of the representation of A on the Hilbert space H, in the sense that there are no non-trivial *closed* linear subspaces of H that are invariant under the representation. While these concepts for a representation of a C*-algebra A on a Hilbert space H are equivalent, the proof that this is the case is non-trivial [8] .

One situation in which these concepts do coincide provides an interesting, and fundamental, characterisation of discrete von Neumann factor quantales:

Theorem 5.1. *A discrete von Neumann quantale Q is a factor if, and only if, the canonical representation*

$$\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$$

is algebraically irreducible.

Proof. Recall that the homomorphism is that defined by assigning to each element $a \in Q$ the sup-preserving mapping

$$\mu_Q(a) : \mathrm{R}(Q) \to \mathrm{R}(Q)$$

that assigns to each element $b \in \mathrm{R}(Q)$ the element $a^* \& b \in \mathrm{R}(Q)$. That this mapping is a homomorphism at all is known to be equivalent to the condition on the Gelfand quantale Q that it is indeed a von Neumann quantale [15] . It has already been remarked that the homomorphism is an isomorphism exactly if the von Neumann quantale Q is a Hilbert quantale [15] . It will now be shown that the representation is irreducible exactly if Q is a von Neumann factor quantale, from which we may then deduce the assertion concerning algebraic irreducibility.

To see this, we recall first that a representation

$$\varphi : Q \to \mathscr{Q}(S)$$

of an involutive unital quantale Q on an orthocomplemented sup-lattice being irreducible is equivalent [16] to the homomorphism concerned being *strong*, in the sense that

$$\varphi(1_Q) = 1_{\mathscr{Q}(S)} .$$

In turn, this is equivalent to the homomorphism restricting to a sup-preserving mapping

$$\mathrm{R}(\varphi) : \mathrm{R}(Q) \to \mathrm{R}(\mathscr{Q}(S))$$

from the right-sided elements of Q to the right-sided elements of $\mathscr{Q}(S)$.

It is asserted that, in the case of the canonical representation

$$\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$$

of a von Neumann quantale Q , these conditions are just equivalent to Q being a factor quantale. On the one hand, if the representation is irreducible, then observe that the element

$$\mu_Q(a) \in \mathscr{Q}(\mathrm{R}(Q))$$

assigned by the canonical homomorphism to any right-sided element $a \in \mathrm{R}(Q)$ is given by

$$b\,\mu_Q(a) = a^* \& b$$

for each $b \in \mathrm{R}(Q)$. However, since $a \in \mathrm{R}(Q)$ implies $a^* \in \mathrm{L}(Q)$, one has that $a^* \& b \in \mathrm{I}(Q)$. Hence,

$$b\,\mu_Q(a) = \begin{cases} 1_Q & \text{unless} \\ 0_Q & b \le a^{\perp} \end{cases} ,$$

for each $b \in \mathrm{R}(Q)$, since $b \le a^{\perp}$ if, and only if, $a^* \& b = 0_Q$. Hence, the sup-preserving mapping

$$\mathrm{R}(\varphi) : \mathrm{R}(Q) \to \mathrm{R}(\mathscr{Q}(\mathrm{R}(Q)))$$

is exactly the canonical isomorphism from the orthocomplemented sup-lattice $\mathrm{R}(Q)$ to that of right-sided elements of the Hilbert quantale $\mathscr{Q}(\mathrm{R}(Q))$. In other words, the canonical homomorphism

$$\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$$

is what we shall refer to as a *right isomorphism* of involutive unital quantales. In particular, the complete Boolean algebra $\mathrm{I}(Q)$ of two-sided elements of Q is isomorphic to that of two-sided elements of the Hilbert quantale $\mathscr{Q}(\mathrm{R}(Q))$, which is exactly $\{0_{\mathscr{Q}(\mathrm{R}(Q))} , 1_{\mathscr{Q}(\mathrm{R}(Q))}\}$. Thus, Q is necessarily a von Neumann factor quantale.

Conversely, if Q is a von Neumann factor quantale, then the sup-preserving mapping

$$\mu_Q(1_Q) : \mathrm{R}(Q) \to \mathrm{R}(Q)$$

assigned to the top element $1_Q \in Q$ maps an element $b \in \mathrm{R}(Q)$ to the element

$$b\,\mu_Q(1_Q) = 1_Q^* \& b = 1_Q \& b$$

which is evidently two-sided. Hence, observing that $b \le 1_Q \& b$ implies that $1_Q \& b = 0_Q$ only if $b = 0_Q$, we have that

$$b\,\mu_Q(1_Q) = \begin{cases} 1_Q & \text{unless} \\ 0_Q & b = 0_Q \end{cases} ,$$

which is exactly the top element $1_{\mathscr{Q}(\mathrm{R}(Q))} \in \mathscr{Q}(\mathrm{R}(Q))$. The canonical homomorphism is therefore irreducible.

Finally, in the event that Q is a discrete von Neumann quantale, we assert that

$$\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$$

being irreducible implies that it is algebraically irreducible. For, since the canonical homomorphism is a right isomorphism, it is also a left isomorphism. Recalling that the algebraic irreducibility of the representation is equivalent to each atom $x \in S$ being a cyclic generator for the representation, suppose given atoms $x, y \in \mathrm{R}(Q)$, and consider the element

$$\kappa_y \in \mathrm{L}(\mathscr{Q}(\mathrm{R}(Q)))$$

defined by

$$b\,\kappa_y = \begin{cases} y & \text{unless} \\ 0_Q & b = 0_Q \end{cases} .$$

Then, since the representation is a left isomorphism, there exists a unique $a \in \mathrm{L}(Q)$ for which

$$\mu_Q(a) = \kappa_y \ ,$$

for which then

$$x\,\mu_Q(a) = y \ .$$

The representation is therefore algebraically irreducible (and the converse is clear). Hence, the discrete von Neumann quantale Q is a factor if, and only if, the canonical representation

$$\mu_Q : Q \rightarrow \mathcal{Q}(\mathrm{R}(Q))$$

is algebraically irreducible. $\square$

Although stated here for the case of a discrete von Neumann quantale, this result specialises that holding more generally for any von Neumann quantale Q in terms of the irreducibility of the canonical representation

$$\mu_Q : Q \rightarrow \mathcal{Q}(\mathrm{R}(Q)) \ ,$$

noted independently by Paseka and Rosický [17] . It may be remarked further that the concept of a strong homomorphism considered there is motivated in part by that of the irreducibility of a representation introduced in [16] . At a later point we shall observe a more intrinsically categorical characterisation of strong homomorphisms in this context, together with a strengthening of this concept that plays an analogous rôle with respect to algebraically irreducible representations.

It may be recalled that discrete von Neumann factor quantales arise naturally by considering the central projections $p \in Q$ of a discrete von Neumann quantale Q associated with projections $q \in \mathrm{Min}(Q)$ that are atoms of the complete atomic Boolean algebra $\mathrm{I}(Q)$ of two-sided elements of Q . Consider therefore such a central projection $p \in Q$ associated with an atom $q \in \mathrm{I}(Q)$ of a discrete von Neumann quantale Q . Observe that the canonical homomorphism

$$\pi_p : Q \rightarrow Q_p$$

to the discrete von Neumann factor quantale Q_p obtained by localising at $p \in Q$, obtained by assigning to each element $a \in Q$ the element $a\&p \in Q_p$, is surjective. Hence, its composite with the canonical representation

$$\mu_{Q_p} : Q_p \rightarrow \mathcal{Q}(\mathrm{R}(Q_p))$$

of the discrete von Neumann factor quantale Q_p yields an irreducible representation of Q on the atomic orthocomplemented sup-lattice $\mathrm{R}(Q_p)$.

Moreover, observe that the elements of $\mathrm{R}(Q_p)$ are exactly those elements $a\&p \in Q$ for which $a\&p = a\&p\&q = a\&p\&p\&1_Q = a\&p\&1_Q = a\&q$ by the relationship $p\&1_Q = q$ that associates $p \in Q$ with $q \in \mathrm{Min}(Q)$. In particular, since $q \in \mathrm{I}(Q)$ is two-sided, it follows that $a \in Q$ may be taken to be right-sided. Hence, the atomic orthocomplemented sup-lattice $\mathrm{R}(Q_p)$ in fact depends only on the element $q \in \mathrm{Min}(Q)$, rather than the particular central projection $p \in Q$ associated with it. Denoting now by

$$\mathrm{R}(Q)_q = \{\, a\&q \in \mathrm{R}(Q) \mid a \in \mathrm{R}(Q) \,\}$$

this particular sup-lattice, we may observe further that the homomorphism

$$\varphi_q : Q \rightarrow \mathcal{Q}(\mathrm{R}(Q)_q)$$

defined above is also independent of the particular $p \in Q$ associated with $q \in \mathrm{Min}(Q)$. For to each $a \in Q$ it assigns the sup-preserving mapping that maps $b\&q \in \mathrm{R}(Q)_q$ to $(a\&p)^*\&(b\&q) \in \mathrm{R}(Q)_q$. But $(a\&p)^*\&(b\&q) = p\&a^*\&b\&q = a^*\&b\&p\&q = a^*\&b\&q$ by the centrality of the projection $p \in Q$ and the fact that $p\&q = p\&1_Q\&q = q\&q = q$.

Since each projection $q \in \mathrm{Min}(Q)$ of a discrete von Neumann quantale is generated by a central projection of Q , we now have the following:

Corollary 5.2. *For each projection* $q \in \mathrm{Min}(Q)$ *of a discrete von Neumann quantale* Q *, the canonical representation*

$$\varphi_q : Q \rightarrow \mathcal{Q}(\mathrm{R}(Q)_q)$$

is algebraically irreducible. $\square$

Consider now for any discrete von Neumann quantale Q the homomorphism

$$\varphi : Q \rightarrow \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q)$$

obtained by taking the product of the canonical representations

$$\varphi_q : Q \rightarrow \mathcal{Q}(\mathrm{R}(Q)_q)$$

indexed by the atoms $q \in \mathrm{Min}(Q)$. Evidently, this is just the homomorphism obtained by choosing a central decomposition $(p_q)_{q \in \mathrm{Min}(Q)}$ of the unit of the discrete von Neumann quantale Q and taking the composite of the isomorphism

$$Q \rightarrow \prod_{q \in \mathrm{Min}(Q)} Q_{p_q}$$

thereby determined with the homomorphism

$$\prod_{q \in \mathrm{Min}(Q)} Q_{p_q} \rightarrow \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q_{p_q}))$$

given by the canonical representation in each component. Since each of these is a right isomorphism, one has the following:

Corollary 5.3. *For any discrete von Neumann quantale* Q *, there is a canonical right isomorphism*

$$\varphi : Q \rightarrow \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q)$$

into the product of discrete Hilbert quantales obtained by localising at each $q \in \mathrm{Min}(Q)$. □

We note once again at this point that this homomorphism is independent of the choice of central decomposition of the unit, as the notation suggests.

6. Spatial Quantales

Recalling that the fundamental insight of Giles and Kummer was that the non-commutative topology defined by a C*-algebra A was represented by the embedding of the closed right ideals of A in the weakly closed right ideals of the atomic von Neumann algebra B along the weakly dense canonical embedding

$$\psi : A \to B \, ,$$

we observe that this is represented at the level of quantales by the fact that the homomorphism

$$\mathrm{Max}\, \psi_w : \mathrm{Max}\, A \to \mathrm{Max}_w\, B$$

from the spectrum of A to the weak spectrum of B restricts on right sides to an embedding. With this in mind, we introduce the following:

Definition. By a *right embedding* of an involutive unital quantale X in a discrete von Neumann quantale Q will be meant a homomorphism

$$\psi : X \to Q$$

of involutive unital quantales that restricts to an embedding of the sup-lattice $\mathrm{R}(X)$ of right-sided elements of X into the sup-lattice $\mathrm{R}(Q)$ of right-sided elements of Q .

Of course, there are two aspects of this condition on a homomorphism of involutive unital quantales. The first is that it must restrict to a homomorphism of sup-lattices of right-sided elements, and the second that this restriction must be an embedding. The first of these conditions is exactly that the homomorphism maps the top element of X to the top element of Q , the condition that we refer to as the homomorphism being strong. It is the second condition that is then the embedding requirement that we need to impose.

Beyond that condition, we also wish to impose one which ensures that the points of the discrete von Neumann quantale into which the involutive unital quantale is right embedded arise from the quantale that we are considering. Observing that the property of a homomorphism

$$\varphi : X \to Q$$

being *strong* in the sense that

$$\varphi(1_X) = 1_Q$$

is exactly the condition that any irreducible representation of Q yields by composition an irreducible representation of X , we make the following:

Definition. By an *algebraically strong* homomorphism

$$\psi : X \to Q$$

from an involutive unital quantale X to an involutive unital quantale Q will be meant a homomorphism of involutive unital quantales with the property that for any algebraically irreducible representation

$$\varphi : Q \to \mathcal{Q}(S)$$

of Q on an atomic orthocomplemented sup-lattice S , the homomorphism

$$\varphi \circ \psi : X \to Q \to \mathcal{Q}(S)$$

is an algebraically irreducible representation of X on S .

In other words, an algebraically strong homomorphism is exactly one that maps points to points. That this is not the case for any homomorphism is not a cause for concern within the quantised world in which pure states are not necessarily mapped to pure states. With these preliminaries, we may now restate the concept of spatiality of an involutive unital quantale:

Definition. An involutive unital quantale X will be said to be *spatial* provided that it admits an algebraically strong right embedding

$$\psi : X \to Q$$

into a discrete von Neumann quantale Q .

An immediate consequence is that any involutive unital quantale X that is spatial is necessarily a Gelfand quantale, simply because the condition is stated in terms of right-sided elements, and is preserved, and hence reflected, under the right embedding into the discrete von Neumann quantale Q , in which it is satisfied.

As a first technical manoeuvre, providing a way in which we can work with the concept of a homomorphism being algebraically strong, we note the following:

Proposition 6.1. *For any discrete von Neumann quantale* Q *, any algebraically irreducible representation*

$$\varphi : \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q) \to \mathcal{Q}(S)$$

is equivalent to the canonical projection

$$\pi_r : \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q) \to \mathcal{Q}(\mathrm{R}(Q)_r)$$

for some unique $r \in \mathrm{Min}(Q)$.

Proof. To simplify notation, throughout this proof we shall write Q_{dis} for the discrete von

Neumann quantale

$$\prod_{q \in \mathrm{Min}(Q)} \mathscr{Q}(\mathrm{R}(Q)_q)$$

with which we are concerned. For each $q \in \mathrm{Min}(Q)$, denote by

$$p_q \in Q_{\mathrm{dis}}$$

the element of this product quantale whose rth component is the unit of $\mathscr{Q}(\mathrm{R}(Q)_r)$ and whose other components are the zero elements of the respective discrete Hilbert quantales. Observe that one has that

$$\bigvee_{q \in \mathrm{Min}(Q)} p_q = e_{Q_{\mathrm{dis}}} ,$$

the unit element of the product quantale. Observe further that, by the algebraic irreducibility of the representation

$$\varphi : Q_{\mathrm{dis}} \to \mathscr{Q}(S) ,$$

choosing an atom $x \in S$ there exists an element $a \in Q_{\mathrm{dis}}$ for which $x\varphi(a) = x$. Noting that $a = a \& e_{Q_{\mathrm{dis}}} = a \& \bigvee_{q \in \mathrm{Min}(Q)} p_q = \bigvee_{q \in \mathrm{Min}(Q)} a \& p_q$, we then have that $x = x\varphi(a) = \bigvee_{q \in \mathrm{Min}(Q)} x\varphi(a)\varphi(p_q) = \bigvee_{q \in \mathrm{Min}(Q)} x\varphi(p_q)$, observing in passing that we have had to argue in this way because a homomorphism of involutive unital quantales is required only to map the unit element above, rather than actually to, the unit element of the quantale. Thus, we have that $x\varphi(p_q) \le x$, hence, by the atomicity of $x \in S$, that $x\varphi(p_r) = x$ for at least one $r \in \mathrm{Min}(Q)$. It is asserted that this $r \in \mathrm{Min}(Q)$ is unique. For suppose that $x\varphi(p_{r'}) = x$ for $r' \in \mathrm{Min}(Q)$. Then $x\varphi(p_r \& p_{r'}) = x\varphi(p_r)\varphi(p_{r'}) = x\varphi(p_{r'}) = x$ implies $r = r'$, since by observation we have that $p_r \& p_{r'} = 0_{Q_{\mathrm{dis}}}$ (and hence that $\varphi(p_r \& p_{r'}) = 0_{\mathscr{Q}(S)}$) unless $r = r'$. Hence, there is a unique $r \in \mathrm{Min}(Q)$ for which

$$x\varphi(p_r) = x .$$

Although this element $r \in \mathrm{Min}(Q)$ might appear to depend on the choice of the atom $x \in S$, this is in fact not the case. For, given any atom $y \in S$, by the above argument there exists a unique $s \in \mathrm{Min}(Q)$ for which

$$y\varphi(p_s) = y ,$$

while by the algebraic irreducibility of the representation there exists an element $a \in Q_{\mathrm{dis}}$ for which

$$x\varphi(a) = y .$$

Hence, $x\varphi(p_r \& a \& p_s) = x\varphi(p_r)\varphi(a)\varphi(p_s) = y$. However, by the centrality of the projections concerned, $p_r \& a \& p_s = a \& p_r \& p_s = 0_{Q_{\mathrm{dis}}}$ unless $r = s$. Hence, there is a unique $r \in \mathrm{Min}(Q)$ for which

$$x\varphi(p_r) = x$$

for each atom $x \in S$, from which it follows that $\varphi(p_r) \in \mathscr{Q}(S)$ is exactly the unit element of the discrete Hilbert quantale $\mathscr{Q}(S)$.

For this unique $r \in \mathrm{Min}(Q)$ it follows that there is a canonical factorisation

$$\begin{tikzcd}
Q_{\mathrm{dis}} \arrow[r] \arrow[dr] & \mathscr{Q}(\mathrm{R}(Q)_r) \arrow[d, dotted] \\
 & \mathscr{Q}(S)
\end{tikzcd}$$

of the algebraically irreducible representation through the projection onto the rth discrete Hilbert quantale. For this to be the case, it suffices to verify that it factorises as a mapping. To see this, observe that $\pi_r(a) = \pi_r(b)$ implies $a \& p_r = b \& p_r$. Then $\varphi(a) = \varphi(a) \& e_{\mathscr{Q}(S)} = \varphi(a) \& \varphi(p_r) = \varphi(a \& p_r) = \varphi(b \& p_r) = \varphi(b) \& \varphi(p_r) = \varphi(b) \& e_{\mathscr{Q}(S)} = \varphi(b)$. Hence, the factorisation homomorphism

$$\mathscr{Q}(\mathrm{R}(Q)_r) \to \mathscr{Q}(S)$$

is well-defined. However, as remarked earlier, any algebraically irreducible representation of a discrete Hilbert quantale is equivalent to the identity representation, giving that the factorisation homomorphism is an isomorphism of discrete Hilbert quantales. Hence, in turn, the algebraically irreducible representation

$$\varphi : Q_{\mathrm{dis}} \to \mathscr{Q}(S)$$

is equivalent to the algebraically irreducible representation

$$\pi_r : Q_{\mathrm{dis}} \to \mathscr{Q}(\mathrm{R}(Q)_r)$$

for a unique $r \in \mathrm{Min}(Q)$, as asserted. □

To which we note the following:

COROLLARY 6.2. *For any discrete von Neumann quantale* Q *, the canonical homomorphism*

$$Q \to \prod_{q \in \mathrm{Min}(Q)} \mathscr{Q}(\mathrm{R}(Q)_q)$$

is an algebraically strong right isomorphism.

Proof. The homomorphism has already been remarked to be a right isomorphism. We now show that it is algebraically strong. Maintaining the notation of the preceding proof, suppose that

$$\varphi : Q_{\mathrm{dis}} \to \mathscr{Q}(S)$$

is an algebraically irreducible representation of the product quantale $\prod_{q \in \mathrm{Min}(Q)} \mathscr{Q}(\mathrm{R}(Q)_q)$ on an atomic orthocomplemented sup-lattice S . Then, by the above, it admits a canonical factorisation

$$\begin{tikzcd}
Q_{\mathrm{dis}} \arrow[r] \arrow[dr] & \mathscr{Q}(\mathrm{R}(Q)_r) \arrow[d, dotted] \\
 & \mathscr{Q}(S)
\end{tikzcd}$$

through the projection $\pi_r : Q_{\text{dis}} \to \mathcal{Q}(\mathrm{R}(Q)_r)$ for a unique $r \in \mathrm{Min}(Q)$ by way of an isomorphism of discrete Hilbert quantales. Composing with the canonical right isomorphism

$$Q \to Q_{\text{dis}}$$

from the discrete von Neumann quantale Q to the product of the discrete Hilbert quantales which it determines, one obtains a representation

$$Q \to Q_{\text{dis}} \to \mathcal{Q}(S)$$

which is equivalent to the canonical algebraically irreducible representation

$$Q \to \mathcal{Q}(\mathrm{R}(Q)_r)$$

obtained by localisation at the element $r \in \mathrm{Min}(Q)$. The representation is therefore also algebraically irreducible, and hence the canonical homomorphism

$$Q \to \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q)$$

is seen to be an algebraically strong right isomorphism of discrete von Neumann quantales. $\square$

Recalling that we have denoted by

$$\varphi_q : Q \to \mathcal{Q}(\mathrm{R}(Q)_q)$$

the algebraically irreducible representation of Q on the atomic orthocomplemented sup-lattice obtained by localising $\mathrm{R}(Q)$ at each $q \in \mathrm{Min}(Q)$, observe that the canonical homomorphism

$$Q \to \prod_{q \in \mathrm{Min}(Q)} \mathcal{Q}(\mathrm{R}(Q)_q)$$

being a right isomorphism implies that for any $a, a' \in \mathrm{R}(Q)$ one has that $a = a'$ if, and in this case only if, $\varphi_q(a) = \varphi_q(a')$ for each $q \in \mathrm{Min}(Q)$, motivating the following:

DEFINITION. An involutive unital quantale X will be said to have *enough points* provided that for any distinct $a, a' \in \mathrm{R}(X)$ there exists an algebraically irreducible representation

$$\varphi : X \to \mathcal{Q}(S)$$

on an atomic orthocomplemented sup-lattice S for which $\varphi(a), \varphi(a') \in \mathrm{R}(\mathcal{Q}(S))$ are distinct.

With these preliminaries, we may now prove:

THEOREM 6.3. *An involutive unital quantale X is spatial if, and only if, X has enough points.*

Proof. Suppose that the involutive unital quantale X is spatial. Then there exists an algebraically strong right embedding

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$$\psi : X \to Q$$

into a discrete von Neumann quantale Q. Suppose that $a, a' \in \mathrm{R}(X)$ are distinct right-sided elements of X. Hence, since the homomorphism into Q is an algebraically strong right embedding, one has that $\psi(a), \psi(a') \in \mathrm{R}(Q)$ are distinct right-sided elements of Q. By the above remarks, there exists an element $q \in \mathrm{Min}(Q)$ such that the algebraically irreducible representation

$$\varphi_q : Q \to \mathcal{Q}(\mathrm{R}(Q)_q)$$

maps $\psi(a), \psi(a') \in \mathrm{R}(Q)$ to distinct right-sided elements of the Hilbert quantale $\mathcal{Q}(\mathrm{R}(Q)_q)$. However, it follows immediately that its composite

$$\varphi_q \circ \psi : X \to Q \to \mathcal{Q}(\mathrm{R}(Q)_q)$$

with the algebraically strong right embedding from X to Q is an algebraically irreducible representation of the involutive unital quantale X on the atomic orthocomplemented sup-lattice $\mathrm{R}(Q)_q$ that maps the right-sided elements $a, a' \in \mathrm{R}(Q)$ to distinct right-sided elements of the Hilbert quantale $\mathcal{Q}(\mathrm{R}(Q)_q)$. Hence, X has enough points.

Conversely, suppose that the involutive unital quantale X has enough points. Choose a family of algebraically irreducible representations

$$\psi_i : X \to \mathcal{Q}(S_i)$$

of X on atomic orthocomplemented sup-lattices S_i indexed by a set I that is complete in the sense that any algebraically irreducible representation of X on an atomic orthocomplemented sup-lattice S is equivalent to one of these. It may be remarked in passing that it is indeed straightforward to show that the equivalence classes of algebraically irreducible representations of any involutive unital quantale X form a set. Considering the homomorphism

$$\psi : X \to \prod_{i \in I} \mathcal{Q}(S_i)$$

determined by this family of algebraically irreducible representations, it is immediate that it is a right embedding into a discrete von Neumann quantale, by the assumption that X has enough points, and the fact that any product of discrete Hilbert quantales is a discrete von Neumann quantale. That the homomorphism is algebraically strong is evident, for given any algebraically irreducible representation

$$\varphi : \prod_{i \in I} \mathcal{Q}(S_i) \to \mathcal{Q}(S)$$

of the product quantale on an atomic orthocomplemented sup-lattice S, there is by the above proposition a unique index $j \in I$ for which the representation is equivalent to that given by the projection

$$\pi_j : \prod_{i \in I} \mathcal{Q}(S_i) \to \mathcal{Q}(S_j) .$$

Hence, its composite

$$\varphi \circ \psi : X \to \prod_{i \in I} \mathcal{Q}(S_i) \to \mathcal{Q}(S)$$

is equivalent, by the definition of the homomorphism concerned, to the composite

$$\pi_j \circ \psi : X \to \prod_{i \in I} \mathcal{Q}(S_i) \to \mathcal{Q}(S_j) \ ,$$

which is exactly the algebraically irreducible representation

$$\psi_j : X \to \mathcal{Q}(S_j) \ .$$

Hence, the homomorphism

$$\psi : X \to \prod_{i \in I} \mathcal{Q}(S_i)$$

is an algebraically strong right embedding of the involutive unital quantale X into a discrete von Neumann quantale. In other words, the involutive unital quantale X is indeed spatial. □

As a consequence of the proof, we have the following:

COROLLARY 6.4. *An involutive unital quantale X is spatial if, and only if, it admits an algebraically strong right embedding*

$$\psi : X \to Q$$

into a product of discrete Hilbert quantales. □

7. CONCLUSIONS

Our first observation is that this leads as a natural consequence to the following:

DEFINITION. By a *quantal space* (X, τ_X) will be meant a Gelfand quantale X together with an algebraically strong right embedding

$$\tau_X : X \to Q_X$$

into a product Q_X of discrete Hilbert quantales. The homomorphism τ_X will be referred to as the *quantal topology* of the quantal space (X, τ_X) .

Of course, any Gelfand quantale X that is spatial may be considered as a quantal space when taken together with its *canonical topology*

$$\tau_X : X \to \prod_{x \in \mathrm{Points}(X)} \mathcal{Q}(S_x)$$

obtained by taking a representative from each equivalence class of points of X . The product is therefore indexed by the set

$$\mathrm{Points}(X)$$

of equivalence classes of points of X , and is determined by the family $(S_x)_{x \in \mathrm{Points}(X)}$ of atomic orthocomplemented sup-lattices on which the corresponding algebraically irreducible representations of X are defined. Indeed, this canonical topology may at least be considered for any involutive unital quantale X , yielding a quantal space exactly if the involutive unital quantale X is spatial.

In the case of a locale L considered as an involutive unital quantale with respect to the identity involution, one has that L is spatial as an involutive unital quantale exactly if it is spatial as a locale. As a consequence, any quantal space (X, τ_X) for which X is a locale consists of an embedding

$$\tau_X : X \to B_X$$

of the locale X into a complete atomic Boolean algebra B_X isomorphic to the power set of the set of points of the locale X .

More generally, given any quantal space (X, τ_X) , observe that the algebraically strong right embedding

$$\tau_X : X \to Q_X$$

restricts to an embedding

$$\mathrm{I}(\tau_X) : \mathrm{I}(X) \to \mathrm{I}(Q_X)$$

from the locale $\mathrm{I}(X)$ of two-sided elements of the Gelfand quantale X into the complete atomic Boolean algebra $\mathrm{I}(Q_X)$ of two-sided elements of the discrete von Neumann quantale Q_X . Hence, any quantal space (X, τ_X) has an underlying classical topological space, of which the set of points is exactly $\mathrm{Points}(X)$ with topology given by the locale $\mathrm{I}(X)$ of two-sided elements of X .

Of course, a homomorphism

$$\varphi : X \to Y$$

of involutive unital quantales will not in general preserve points, so this construction of the topological space underlying a quantal space cannot be expected to be functorial on the category of quantal spaces implied by the following:

DEFINITION. By a *homomorphism*

$$(\varphi, \tau_\varphi) : (X, \tau_X) \to (Y, \tau_Y)$$

of quantal spaces will be meant a homomorphism

$$\varphi : X \to Y$$

of Gelfand quantales together with a homomorphism

$$\tau_\varphi : Q_X \to Q_Y$$

of discrete von Neumann quantales for which the diagram

$$\begin{tikzcd}
Q_X \arrow[r, "\tau_\varphi"] & Q_Y \\
X \arrow[u, "\tau_X"] \arrow[r, "\varphi"] & Y \arrow[u, "\tau_Y"']
\end{tikzcd}$$

commutes.

In another direction, one would not expect to find a reflection of the category of involutive unital quantales into that of quantal spaces, generalising that of the category of locales into that of topological spaces. It may however be straightforwardly observed that assigning to any quantal space its underlying classical topological space is indeed functorial provided that we consider only those homomorphisms of quantal spaces in which the underlying homomorphism

$$\varphi : X \to Y$$

is algebraically strong. More importantly, one then has the following:

THEOREM 7.1. *For any algebraically strong homomorphism*

$$\varphi : X \to Y$$

from the Gelfand quantale X *of a quantal space* (X, τ_X) *to the Gelfand quantale* Y *of a quantal space* (Y, τ_Y) *, there is a unique homomorphism*

$$\tau_\varphi : Q_X \to Q_Y$$

of discrete von Neumann quantales for which the diagram

$$\begin{tikzcd}
Q_X \arrow[r, "\tau_\varphi"] & Q_Y \\
X \arrow[u, "\tau_X"] \arrow[r, "\varphi"] & Y \arrow[u, "\tau_Y"']
\end{tikzcd}$$

commutes.

Proof. First, we need to establish some notation. For each $x \in \text{Points}(X)$, we write

$$\tau_x : X \to \mathcal{Q}(S_x)$$

for the algebraically irreducible representation of X indexed by $x \in \text{Points}(X)$ of the quantal space (X, τ_X) to yield the quantal topology

$$\tau_X : X \to \prod_{x \in \text{Points}(X)} \mathcal{Q}(S_x)$$

as the product of these representations. Similarly, the quantal topology of the quantal space (Y, τ_Y) arises from algebraically irreducible representations of Y which we shall denote by

$$\tau_y : Y \to \mathcal{Q}(T_y)$$

for each $y \in \text{Points}(Y)$. Observe that since $\varphi : X \to Y$ is algebraically strong, for each point $y \in \text{Points}(Y)$ the homomorphism

$$\tau_y \circ \varphi : X \to Y \to \mathcal{Q}(T_y)$$

is an algebraically irreducible representation of X, hence has equivalence class a point of X which we shall denote by $\varphi(y) \in \text{Points}(X)$.

Observe that the point $\varphi(y) \in \text{Points}(X)$ thereby obtained indexes a projection

$$\pi_{\varphi(y)} : \prod_{x \in \text{Points}(X)} \mathcal{Q}(S_x) \to \mathcal{Q}(S_{\varphi(y)})$$

of the product quantale Q_X . Considering the representation

$$\pi_{\varphi(y)} \circ \tau_X : X \to Q_X \to \mathcal{Q}(S_{\varphi(y)})$$

of the involutive unital quantale X, which is exactly the algebraically irreducible representation of X indexed by the point $\varphi(y) \in \text{Points}(X)$, observe that by construction this is equivalent to

$$\pi_y \circ \tau_Y \circ \varphi : X \to Y \to Q_Y \to \mathcal{Q}(T_y) \ ,$$

hence determines a unique isomorphism

$$\tau_y : \mathcal{Q}(S_{\varphi(y)}) \to \mathcal{Q}(T_y) \ ,$$

induced by an isomorphism of orthocomplemented sup-lattices, for which the diagram

$$\begin{tikzcd}
\mathcal{Q}(S_{\varphi(y)}) \arrow[r, "\tau_y"] & \mathcal{Q}(T_y) \\
Q_X \arrow[u, "\pi_{\varphi(y)}"] \arrow[r, dashed, "\tau_\varphi"] & Q_Y \arrow[u, "\pi_y"'] \\
X \arrow[u, "\tau_X"] \arrow[r, "\varphi"] & Y \arrow[u, "\tau_Y"']
\end{tikzcd}$$

commutes. Define

$$\tau_\varphi : Q_X \to Q_Y$$

to be the unique homomorphism of discrete von Neumann quantales for which the diagram

$$\begin{tikzcd}
\mathcal{Q}(S_{\varphi(y)}) \arrow[r, "\tau_y"] & \mathcal{Q}(T_y) \\
Q_X \arrow[u, "\pi_{\varphi(y)}"] \arrow[r, "\tau_\varphi"] & Q_Y \arrow[u, "\pi_y"']
\end{tikzcd}$$

commutes for each $y \in \text{Points}(Y)$. Then, by the observation that the projection

$$\pi_y : Q_Y \to \mathcal{Q}(T_y)$$

is surjective, it is also the unique homomorphism such that the diagram

$$\begin{tikzcd}
Q_X \arrow[r, "\tau_\varphi"] & Q_Y \\
X \arrow[u, "\tau_X"] \arrow[r, "\varphi"] & Y \arrow[u, "\tau_Y"']
\end{tikzcd}$$

commutes, as asserted. □

Indeed, the construction of the homomorphism whose existence is established leads us to the following:

Definition. A homomorphism

$$\tau : Q_X \to Q_Y$$

of products of discrete Hilbert quantales $\prod_{x\in X}\mathscr{Q}(S_x)$, $\prod_{y\in Y}\mathscr{Q}(T_y)$ respectively will be said to be a *local equivalence* provided that for each projection

$$\pi_y : Q_Y \to \mathscr{Q}(T_y)$$

of the product Q_Y there exists a unique projection

$$\pi_x : Q_X \to \mathscr{Q}(S_x)$$

to which the representation

$$\pi_y \circ \tau : Q_X \to Q_Y \to \mathscr{Q}(S_y)$$

is equivalent.

An immediate consequence of the construction of the preceding theorem is the following:

Corollary 7.2. *For any homomorphism*

$$(\varphi, \tau_\varphi) : (X, \tau_X) \to (Y, \tau_Y)$$

of quantal spaces, the homomorphism $\varphi : X \to Y$ *of Gelfand quantales is algebraically strong if, and only if, the homomorphism* $\tau_\varphi : Q_X \to Q_Y$ *of products of discrete Hilbert quantales is a local equivalence.* □

More generally, by restricting to the category of involutive unital quantales and algebraically strong homomorphisms, we may obtain a reflection into the category of quantal spaces, in the following sense:

Corollary 7.3. *For any involutive unital quantale* X *, there is a quantal space* $(\widetilde{X}, \tau_{\widetilde{X}})$ *together with a homomorphism*

$$\varepsilon_X : X \to \widetilde{X}$$

of involutive unital quantales such that any algebraically strong homomorphism

$$\varphi : X \to Y$$

from X *to the quantale of a quantal space* (Y, τ_Y) *factors uniquely through an algebraically strong homomorphism*

$$\psi : \widetilde{X} \to Y$$

of involutive unital quantales for which the diagram

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[d, "\psi"] \\ & Y \end{tikzcd}$$

commutes.

Proof. Let $\tau_X : X \to Q_X$ be the canonical topology on the involutive unital quantale X , defined by taking the product

$$\tau_X : X \to \prod_{x\in \mathrm{Points}\,(X)}\mathscr{Q}(S_x)$$

of a family of algebraically irreducible representations of X representing the points of X .

Observe that this homomorphism is necessarily algebraically strong, since each algebraically irreducible representation of the product Q_X of discrete Hilbert quantales factors through a unique projection of the product. Consider the involutive unital quantale $X/\sim$ obtained by taking the quotient with respect to the least congruence on X for which

$$\tau_X(a) = \tau_X(a') \quad \text{implies} \quad a \sim a'$$

for all right-sided elements $a, a' \in \mathrm{R}\,(X)$. Writing the quotient homomorphism thereby determined by

$$\varepsilon_X : X \to \widetilde{X} ,$$

observe that the homomorphism

$$\tau_{\widetilde{X}} : \widetilde{X} \to Q_X$$

induced by $\tau_X : X \to Q_X$ is necessarily a right embedding, by construction of the quotient concerned. It is also algebraically strong, by the fact that τ_X factors through $\tau_{\widetilde{X}}$. Hence, $(\widetilde{X}, \tau_{\widetilde{X}})$ is indeed a quantal space.

Now, suppose given any homomorphism

$$\varphi : X \to Y$$

of involutive unital quantales from X to the quantale Y of a quantal space (Y, τ_Y) . Proceeding exactly as in the proof of the preceding proposition, we may find a unique homomorphism

$$\tau_\varphi : Q_X \to Q_Y$$

of discrete von Neumann quantales, for which the diagram

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[d, "\varphi"'] & Q_X \arrow[d, "\tau_\varphi"] \\ Y \arrow[r, "\tau_Y"] & Q_Y \end{tikzcd}$$

commutes, since the construction depends only on $\tau_X : X \to Q_X$ being the canonical topology. Observe that the homomorphism

$$\varphi : X \to Y$$

necessarily factors through the quotient homomorphism

$$\varepsilon_X : X \to \widetilde{X} ,$$

by the commutativity of the above diagram and the fact that

$$\tau_Y : Y \to Q_Y$$

is a right embedding. Hence, there is a unique homomorphism

$$\psi : \widetilde{X} \to Y$$

of involutive unital quantales for which the diagram

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[r, "\tau_{\widetilde{X}}"] \arrow[d, "\psi"] & Q_X \arrow[d, "\tau_\varphi"] \\
 & Y \arrow[r, "\tau_Y"] & Q_Y
\end{tikzcd}$$

commutes. Moreover, by the existence of the factorisation, the homomorphism

$$\psi : \widetilde{X} \to Y$$

is evidently algebraically strong, as asserted. □

Finally, returning to the case of C*-algebras and the motivating insight of Giles and Kummer, we note that the spectrum

$$\mathrm{Max}\ A$$

of any C*-algebra A is a spatial quantale. In particular, the homomorphism

$$\tau_A : \mathrm{Max}\ A \to Q_{\mathrm{Max}\ A}$$

from the spectrum $\mathrm{Max}\ A$ into the product of the Hilbert quantales determined by the irreducible representations of A on Hilbert space is a quantal topology. Moreover, it may be shown directly that the canonical homomorphism

$$\mathrm{Max}\ \psi : \mathrm{Max}\ A \to \mathrm{Max}_w\ B$$

from the spectrum of A into the weak spectrum $\mathrm{Max}_w\ B$ induced by the weakly dense embedding

$$\psi : A \to B$$

of the C*-algebra A into the atomic von Neumann algebra B determined by the irreducible representations of A on Hilbert space is an algebraically strong right embedding of $\mathrm{Max}\ A$ into the discrete von Neumann quantale $\mathrm{Max}_w\ B$.

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