Quantales

A complete sup-lattice Q together with an associative product & satisfying the distributive laws

$$a\&(\vee_i b_i) = \vee_i a\&b_i \quad \text{and} \quad (\vee_i b_i)\&a = \vee_i b_i\&a$$

for all $a, b_i \in Q$. The name ‘quantale’ was introduced by C.J.Mulvey [1] to provide a non-commutative extension of the concept of *locale*. The intention was to develop the concept of non-commutative topology introduced by R.Giles and H.Kummer [2], while providing a constructive, and non-commutative, context for the foundations of quantum mechanics and, more generally, non-commutative logic. The observation that the closed right ideals of a C*-algebra form a quantale satisfying the conditions that each element is *right-sided* ($a\&1_Q \leq a$) and *idempotent* ($a\&a = a$) led certain authors to restrict the term quantale to mean only quantales of this kind [3], but the term is now applied only in its original sense. The realisation by J.Rosický [4] that the development of topological concepts such as regularity required additional structure led [5] to the consideration of *involutive quantales* and of the *spectrum* $\mathrm{Max}\,A$ of a C*-algebra A as the quantale of closed linear subspaces of A, together with the operations of join given by closed linear sum, product given by closed linear product of subspaces, and involution by involution within the C*-algebra. The right-sided elements of the spectrum $\mathrm{Max}\,A$ are the closed right ideals of the C*-algebra A (cf. [2,6]). By the existence of approximate units, each element $a \in \mathrm{R}(\mathrm{Max}\,A)$ of the sup-lattice of right-sided elements satisfies the condition that $a\&a^*\&a = a$. By a *Gelfand quantale* Q is meant an involutive unital quantale in which the right-sided (equivalently left-sided) elements satisfy this condition. Generalising an observation of [4], the right-sided elements of any involutive quantale Q may be shown to admit a *pseudo-orthocomplement* defined by $a^{\perp} = \vee_{a^*\&b=0_Q} b$. In any Gelfand quantale Q, the right-sided elements are idempotent, and the two-sided elements form a locale.

Observing that relations on a set X forming a quantale with respect to arbitrary union and composition is applied implicitly by C.A.Hoare and He Jifeng to considering the *weakest prespecification* of a program [7], and noting that the quantale $\mathscr{Q}(X)$ concerned is exactly that of endomorphisms of the sup-lattice $\mathscr{P}(X)$ of subsets of X led to the consideration [8] of the quantale $\mathscr{Q}(S)$ of endomorphisms of any orthocomplemented sup-lattice S, in which the involute α^* of a sup-preserving mapping α is defined by $s\alpha^* = (\vee_{t\alpha \leq s^{\perp}} t)^{\perp}$ for each $s \in S$. In the quantale of relations $\mathscr{Q}(X)$ on a set X, this describes the reverse of a relation in terms of complementation of subsets. Observing that the quantale $\mathscr{Q}(H)$ of endomorphisms of the orthocomplemented sup-lattice of closed linear subspaces of a Hilbert space H provides a motivating example for this quantisation of the calculus of relations, the term *Hilbert quantale* was introduced for any quantale isomorphic to the quantale $\mathscr{Q}(S)$ of an orthocomplemented sup-lattice S. Noting that the *weak spectrum* $\mathrm{Max}_w B$ of a von Neumann algebra B is a Gelfand quantale of which the right-sided elements correspond to the projections of B and on which the right pseudo-orthocomplement corresponds to orthocomplementation of projections, a Gelfand quantale Q is said to be a *von Neumann quantale* provided that $a^{\perp\perp} = a$ for any right-sided element $a \in Q$. For any von Neumann quantale Q, the locale $\mathrm{I}(Q)$ of two-sided elements is a complete Boolean algebra. Any Hilbert quantale Q is a von Neumann quantale, and a von Neumann quantale Q is a Hilbert quantale exactly if the canonical homomorphism $\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$, assigning to each $a \in Q$ the sup-preserving mapping $b \in \mathrm{R}(Q) \mapsto a^*\&b \in \mathrm{R}(Q)$ on the orthocomplemented sup-lattice $\mathrm{R}(Q)$ of right-sided elements of Q, is an isomorphism [8]. Any Hilbert quantale Q is a *von Neumann factor quantale* in the sense that $\mathrm{I}(Q)$ is exactly $\mathbf{2}$. The weak spectrum $\mathrm{Max}_w B$ of a von Neumann algebra B is a factor exactly if B is a factor [9].

A homomorphism $\varphi : Q \to \mathscr{Q}(S)$ from a Gelfand quantale Q to the Hilbert quantale $\mathscr{Q}(S)$ of an orthocomplemented sup-lattice S is said to be a *representation* of Q on S [10]. A representation is said to be *irreducible* provided that $s \in S$ invariant (in the sense that $s\varphi_a \leq s$ for all $a \in Q$) implies $s = 0_S$ or $s = 1_S$. The irreducibility of a representation $\varphi : Q \to \mathscr{Q}(S)$ is equivalent to the homomorphism being *strong*, in the sense that $\varphi(1_Q) = 1_{\mathscr{Q}(S)}$. A homomorphism $Q' \to Q$ of Gelfand quantales is strong exactly if $Q' \to Q \to \mathscr{Q}(S)$ is irreducible whenever $Q \to \mathscr{Q}(S)$ is irreducible. A representation $\varphi : Q \to \mathscr{Q}(S)$ of Q on an atomic orthocomplemented sup-lattice S is said to be *algebraically irreducible* provided that for any atoms $x, y \in S$ there exists $a \in Q$ such that $x\varphi_a = y$. Any algebraically irreducible representation is irreducible: the algebraically irreducible representations are those for which every atom is a *cyclic generator*. An algebraically irreducible representation $\varphi : Q \to \mathscr{Q}(S)$ on an atomic orthocomplemented sup-lattice S is said to be a *point* of the Gelfand quantale Q. The points of the spectrum $\mathrm{Max}\,A$ of a C*-algebra A correspond bijectively to the equivalence classes of irreducible representations of A on Hilbert space [10] (presently subject to a conjecture that every point of $\mathrm{Max}\,A$ is *non-trivial* in the sense that there exists a pure state that maps properly). (For a discussion of the role of pure states in this context, see [10]). In particular, the spectrum $\mathrm{Max}\,A$ is an invariant of the C*-algebra A. It may be noted that the Hilbert quantale $\mathscr{Q}(S)$ of an atomic orthocomplemented sup-lattice has, to within equivalence, a unique point: moreover, the reflection of such a Gelfand quantale into the category of locales is exactly $\mathbf{2}$. In particular, the points of any locale are exactly its points in the sense of the theory of locales.

A von Neumann quantale Q is said to be *atomic* provided that the orthocomplemented sup-lattice $\mathrm{R}(Q)$ of its right-sided elements is atomic. For any atomic von Neumann quantale Q the complete Boolean algebra $\mathrm{I}(Q)$ of two-sided elements is atomic. Moreover, the canonical homomorphism $\mu_Q : Q \to \mathscr{Q}(\mathrm{R}(Q))$ is algebraically irreducible exactly if Q is a von Neumann factor quantale. A Gelfand quantale Q is said to be *discrete* provided that it is an atomic von Neumann quantale that admits a *central decomposition* of the unit $e_Q \in Q$, in the sense that the atoms of the complete Boolean algebra $\mathrm{I}(Q)$ majorise a family of central projections with join $e_Q \in Q$. For any atomic von Neumann algebra B, the weak spectrum $\mathrm{Max}_w B$ is a discrete von Neumann quantale. A locale L is a discrete von Neumann quantale exactly if it is a complete atomic Boolean algebra, hence the power set of its set of points. A homomorphism $X \to Q$ of Gelfand quantales is said to be *algebraically strong* provided that $X \to Q \to \mathscr{Q}(S)$ is algebraically irreducible whenever $Q \to \mathscr{Q}(S)$ is an algebraically irreducible representation of Q on an atomic orthocomplemented sup-lattice S, to be a *right embedding* provided that it restricts to an embedding $\mathrm{R}(X) \to \mathrm{R}(Q)$ of the lattices of right-sided elements, and to be *discrete* if it is an algebraically strong right embedding. A Gelfand quantale X is said to *spatial* provided that it admits a discrete homomorphism $X \to Q$ into a discrete von Neumann quantale Q [11]. For any C*-algebra A, the canonical homomorphism $\mathrm{Max}\,A \to \mathrm{Max}_w B$ of its spectrum $\mathrm{Max}\,A$ into the weak spectrum of its enveloping atomic von Neumann algebra B is discrete, hence $\mathrm{Max}\,A$ is spatial. Similarly, a locale L is spatial as a Gelfand quantale exactly if its canonical homomorphism into the power set of its set of points is discrete. More generally, a Gelfand quantale Q is spatial exactly if it has enough points, in the sense that if $a, b \in \mathrm{R}(Q)$ are distinct then there is an algebraically irreducible representation $\varphi : Q \to \mathscr{Q}(S)$ on an atomic orthocomplemented sup-lattice S such that $\varphi_a, \varphi_b \in \mathrm{R}(\mathscr{Q}(S))$

are distinct [11].

In other important directions, *Girard quantales* have been shown [12] to provide a semantics for non-commutative linear logic, and *Foulis quantales* to generalise the Foulis semigroups of complete orthomodular lattices [13]. The concepts of *quantal set* and of *sheaf* have been introduced [14] for the case of idempotent right-sided quantales, generalising those for any locale. These concepts may be localised [15] to allow the construction of a fibration from which the quantale may be recovered directly. The representation of quantales by quantales of relations has also been examined [16].

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