

# The Stone-Ćech Compactification of Locales, III

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## INTRODUCTION

In this note we take another look, after a number of intervening years, at aspects of the Stone-Ćech compactification  $\beta L$  of a locale  $L$ , this time concerning ourselves with the question of whether  $\beta L$  can be described by a propositional theory formulated *directly* in terms of  $L$ . It should be emphasised that  $\beta L$  can certainly be described by *some* propositional theories. Thus Banaschewski-Mulvey [4] presented it, shortly after establishing the existence by lattice-theoretic means in [3], by that of the notion of maximal ideal in the ring of continuous real-valued functions on  $L$ , and it is implicit in Banaschewski [2] that  $\beta L$  is also given by the propositional theory of a regular Cauchy filter in a suitable uniformity on  $L$  - but these hardly qualify as descriptions *directly* in terms of  $L$ .

In the following, we introduce the notion of *almost prime* completely regular filters on a locale  $L$ , show constructively that these are exactly, in an appropriate sense, the maximal completely regular filters on  $L$ , and establish that the propositional theory of almost prime completely regular filters provides a description of  $\beta L$ . Further, we show that the weak form of  $\beta L$ , the compact regular (rather than completely regular) reflection of  $L$ , and the compact zero-dimensional reflection  $\zeta L$  of a locale  $L$  are given by natural variants of this theory. Finally, we observe that the compactification of a regular locale  $L$  derived from a normal basis of  $L$  also fits into this pattern, and that indeed any compactification of a locale  $L$  can be described by the same kind of propositional theory.

## 1. ALMOST PRIME FILTERS

For basic notions concerning locales we refer to Johnstone [6] or Vickers [10]. As to compactifications, we use the term Stone-Ćech compactification for an arbitrary locale in its

extended meaning as the compact completely regular reflection of  $L$ , which will determine a dense sublocale embedding  $L \rightarrow \beta L$  whenever  $L$  is itself completely regular. Regarding the latter notion, recall that on any locale  $L$  one has the completely below (also: really inside) relation, which here will be denoted by  $v \ll u$ , and  $L$  is called *completely regular* whenever  $u = \bigvee \{v \in L \mid v \ll u\}$  for all  $u \in L$ . Further, we note the following properties of the relation  $\ll$  needed in our context:

$0 \ll 0$  and  $e \ll e$  for the zero (= bottom)  $0$  and the unit (= top)  $e$  of  $L$ ;

if  $u' \leq u$ ,  $u \ll v$ , and  $v \leq v'$  then  $u' \ll v'$ ;

if  $u \ll w$  and  $v \ll w$  then  $u \vee v \ll w$ ;

if  $u \ll v$  and  $u \ll w$  then  $u \ll v \wedge w$ ;

if  $u \ll v$  then  $u \ll w \ll v$  for some  $w$ ;

if  $u \ll v$  then  $v^* \ll u^*$  where  $*$  denotes pseudocomplement;

if  $u \ll v$  then  $u^* \vee v = e$ .

It may be noted that one also has the *rather below* relation  $\prec$  (in terms of which the completely below relation is defined) determined by the last of these conditions.

A filter  $P$  on a locale  $L$  is called *completely regular* if

for any  $u \in P$  there exists  $v \ll u$  in  $P$ .

A completely regular filter  $P$  will be said to be *almost prime* if it is proper, in the sense that

$\neg(0 \in P)$ , and

$u \vee v \in P$  and  $u \ll u'$ ,  $v \ll v'$  implies  $u' \in P$  or  $v' \in P$ .

Note that for a *Boolean* locale  $\ll$  coincides with  $\leq$ , so that every filter is completely regular and almost prime means prime.

In the following, we shall use the notion of maximality of filters introduced by Mulvey [7] for the filters relevant here. Thus, a completely regular filter  $P$  on a locale  $L$  will be called *maximal* if  $P$  is proper and for any completely regular filter  $F \supseteq P$  on  $L$ ,  $u \in F$  implies  $u \in P$  or  $0 \in F$  for any  $u \in L$ . We note that, similarly to results in [7], this coincides with maximality amongst proper completely regular filters in the usual sense of the partial order given by subset inclusion if the logic of our setting satisfies the De Morgan Law, as will be shown at the end of the paper. However, constructively one has:

LEMMA. *A completely regular filter on a locale is maximal completely regular if, and only if, it is almost prime.*

*Proof.* ( $\Rightarrow$ ) We first show, for any maximal completely regular filter  $P$  on a locale  $L$ , that given any completely regular filters  $F$  and  $G$  on  $L$  for which  $F \wedge G \leq P$ , then one has for any  $u \in F$  and  $v \in G$  that  $u \in P$  or  $v \in P$ . For, by maximality, one has that  $u \in P$  or  $0 \in P \vee F$  and  $v \in P$  or  $0 \in P \vee G$ , hence

$$u \wedge v \in P \text{ or } u \in P \text{ and } 0 \in P \vee F \text{ or } 0 \in P \vee G \text{ and } v \in P \text{ or } 0 \in P$$

But, since  $P$  is proper, it follows that  $u \wedge v \in P$  or  $u \in P$  or  $v \in P$ , and consequently also that  $u \in P$  or  $v \in P$ , as claimed.

Now consider any  $u \vee v \in P$  where  $u \ll u'$  and  $v \ll v'$ . Then

$$F = \{x \in L \mid u \ll x\} \text{ and } G = \{y \in L \mid v \ll y\}$$

are completely regular filters such that  $F \wedge G \leq P$ . Hence,  $x \in P$  or  $y \in P$  for any  $x \in F$  and  $y \in G$ , as just shown, and thus in particular  $u' \in P$  or  $v' \in P$ .

( $\Leftarrow$ ) Let  $F$  and  $P$  be completely regular filters on a locale  $L$  such that  $F \geq P$  and  $P$  is almost prime. Now, for any  $u \in F$ , by complete regularity choose  $v \ll u$  in  $F$  and further  $v \ll w \ll x \ll u$ . Then  $x \vee w^* = e \in P$ , while  $x \ll u$  and  $w^* \ll v^*$ , hence  $u \in P$  or  $v^* \in P$ . But the latter implies  $v^* \in F$ , so that  $0 = v \wedge v^* \in F$ . Hence, for any  $u \in F$ , one has that  $u \in P$  or  $0 \in F$ , as desired.  $\square$

REMARK. It may be observed that the proof of the forward implication consists of noting first that any proper completely regular filter  $P$  that is *maximal*, in the sense that

$$F \geq P \text{ implies } u \in P \vee 0 \in F$$

for any completely regular filter  $F$  and any  $u \in F$ , is *prime*, in the sense that

$$F \wedge G \leq P \text{ implies } u \in P \vee v \in P$$

for any completely regular filters  $F, G$  and any  $u \in F, v \in G$ . One then applies this to show that the completely regular filter  $P$  is almost prime. Indeed, it may be seen straightforwardly that the notion of primeness in the lattice of completely regular filters described in this way is exactly equivalent to that of almost primeness introduced above. Evidently, this constructivisation of the concept of primeness is analogous to that of maximality introduced in Mulvey [7]. Indeed, since, as we have already remarked, in the case of a Boolean algebra every filter is completely regular and almost primeness coincides with primeness, the result just proved may evidently be seen as a generalisation to the case of locales of the result proved in Mulvey [7] that in any Boolean algebra the prime filters are exactly the maximal filters.

## 2. STONE-ČECH COMPACTIFICATION

Recall that, classically, the Stone-Čech compactification of a topological space  $X$  may be described as the space  $\beta X$  of maximal completely regular filters on the locale of open sets of  $X$ , together with the embedding  $X \rightarrow \beta X$  obtained by mapping each point of  $X$  to its open neighbourhood filter. This suggests that the Stone-Čech compactification of a locale  $L$  might be described by the following propositional theory, which expresses the notion of an almost prime, and hence maximal, completely regular filter on  $L$  in terms of basic propositions  $u \in P$ , for each  $u \in L$ , and the following axioms:

- (1)  $0 \in P \vdash \text{false}, \text{true} \vdash e \in P$ ;
- (2)  $u \in P \wedge v \in P \vdash u \wedge v \in P$ ;
- (3)  $u \in P \vdash \bigvee \{v \in P \mid v \ll u\}$ ;
- (4)  $u \vee v \in P \vdash u' \in P \vee v' \in P$  whenever  $u \ll u'$  and  $v \ll v'$ .

Denote by  $\mathfrak{M}L$  the Lindenbaum locale of this theory: that is, the locale given by the propositions of the theory, obtained from the basic propositions by finite conjunction and arbitrary disjunction, ordered by provable entailment, modulo provable equivalence within the theory. Since the locale corresponding to the axioms (1) and (2) is isomorphic to  $\mathfrak{D}L$ , the locale of inhabited downsets of  $L$ , of which the elements are those subsets  $U \subseteq L$  such that

$$0 \in U, \text{ and}$$

$$u \in U \text{ and } v \leq u \text{ implies } v \in U,$$

it may be verified straightforwardly that  $\mathfrak{M}L$  may in turn be identified with the sublocale of  $\mathfrak{D}L$  determined by the closure conditions

$$v \in U \text{ for all } v \ll u \text{ implies } u \in U, \text{ and}$$

$$u, v \in U \text{ and } u' \ll u, v' \ll v \text{ implies } u' \vee v' \in U$$

corresponding to the remaining axioms. We observe that the last of these conditions may be described as saying of an inhabited downset  $U$  that it is *almost an ideal*. In order to see that these conditions do what is claimed, it has to be checked that they indeed define a sublocale of  $\mathfrak{D}L$ , and this is best done by verifying straightforwardly the familiar criterion that  $W \rightarrow U$  will satisfy them whenever  $U$  does for any  $W$  and  $U$  in  $\mathfrak{D}L$ .

Regarding notation, we let  $[u]$  stand for the element of  $\mathfrak{M}L$  determined by the principal downset  $\downarrow u = \{v \in L \mid v \leq u\}$ . Note that these  $[u]$  generate  $\mathfrak{M}L$  with respect to arbitrary joins since the  $\downarrow u$  do this in  $\mathfrak{D}L$ . Further,  $[u] \wedge [v] = [u \wedge v]$  because  $\downarrow u \wedge \downarrow v = \downarrow u \wedge v$  and the closure operator involved here is a nucleus. Of course, if  $L$  is completely regular then  $[u] = \downarrow u$ .

We now establish various properties of the locale  $\mathfrak{M}L$ :

$\mathfrak{M}L$  is *completely regular*. For any  $v \ll u$  in  $L$ , let  $v \ll w \ll x \ll u$  so that  $x \ll u$ ,  $w^* \ll v^*$ , and  $x \vee w^* = e$ . This shows, by the second of the above closure conditions, that  $[u] \vee [v^*] = [e]$ , and since  $[v] \wedge [v^*] = [0]$  it follows that  $[v] \prec [u]$ ; further, by the properties of  $\ll$  this implies  $[v] \ll [u]$ . Finally,  $[u] = \bigvee \{[v] \mid v \ll u\}$  by the first closure condition, and this proves complete regularity.

$\mathfrak{M}L$  is *compact*. We use the locale  $\mathfrak{I}L$  of ideals of  $L$  as an auxiliary device. First note that the correspondence  $u \in P \mapsto k(u) = \{v \in L \mid v \ll u\}$  is a model of our theory in  $\mathfrak{I}L$ . For  $k(0) = \downarrow 0$ ,  $k(e) = \downarrow e$ , and  $k(u \wedge v) = k(u) \wedge k(v)$  results from the basic properties of  $\ll$ . Further,  $k(u) = \bigvee \{k(v) \mid v \ll u\}$  is exactly the condition that, for any  $v \ll u$ , there exist  $w$  such that  $v \ll w \ll u$ ;  $\bigvee$  is union here because the set of these  $k(v)$  is updirected. Finally, if  $u \ll u'$  and  $v \ll v'$  then  $s \wedge u \ll u'$ ,  $s \wedge v \ll v'$ , and  $s = (s \wedge u) \vee (s \wedge v)$  for any  $s \leq u \vee v$ , and hence, in particular,  $k(u \vee v) \leq k(u') \vee k(v')$ . As a result, we have a frame homomorphism  $h : \mathfrak{M}L \rightarrow \mathfrak{I}L$  such that  $h([u]) = k(u)$ . This is dense since  $k(u) = \downarrow 0$  implies  $[u] = [0]$ , hence a frame embedding because  $\mathfrak{M}L$  is regular and  $\mathfrak{I}L$  is compact, and this makes  $\mathfrak{M}L$  compact.

The correspondence  $L \rightarrow \mathfrak{M}L$  is *functorial*. Any locale map  $h : M \rightarrow L$  determines a model  $\mu$  in  $\mathfrak{M}L$  of our theory for the locale  $L$  such that

$$\mu(u \in P) = \bigvee \{[h^*(u')] \mid u' \ll u\}$$

for each  $u \in L$ . Hence, the conditions concerning the axioms (1) and (2) follow immediately by the lattice properties of  $\ll$ , and for (3) one has

$$\bigvee \{ \mu(v \in P) \mid v \ll u \} = \bigvee \{ [h^*(v')] \mid v' \ll v \ll u \}$$

since  $\ll$  interpolates. Finally, regarding (4), in order to see that

$$\mu(u \vee v \in P) \leq \mu(u' \in P) \vee \mu(v' \in P)$$

whenever  $u \ll u'$  and  $v \ll v'$  in  $L$ , it has to be shown that for any  $w \ll u \vee v$  that  $h^*(w)$  belongs to

$$S = \bigvee \{ [h^*(u'')] \mid u'' \ll u' \} \vee \bigvee \{ [h^*(v'')] \mid v'' \ll v' \}.$$

For this, let  $u \ll u'' \ll u'$  and  $v \ll v'' \ll v'$ . Then  $w \wedge u \ll u''$  and  $w \wedge v \ll v''$ , hence  $h^*(w \wedge u) \ll h^*(u'')$  and  $h^*(w \wedge v) \ll h^*(v'')$ , and since  $h^*(u''), h^*(v'') \in S$  the same holds for  $h^*(w) = h^*(w \wedge u) \vee h^*(w \wedge v)$  by the properties of  $\mathfrak{M}$ . As a result, we have a locale map

$$\tilde{h} : \mathfrak{M} \rightarrow \mathfrak{M}$$

such that  $h^*([u]) = \bigvee \{ [h^*(u')] \mid u' \ll u \}$ . Further, if  $h = \text{id}_L$  then  $\tilde{h} = \text{id}_{\mathfrak{M}}$  by the properties of  $\mathfrak{M}$ , and for any  $h : M \rightarrow L$  and  $g : L \rightarrow N$  simple calculation shows that  $\tilde{h} \tilde{g}^*([u]) \leq \tilde{g}^*([u])$  which implies  $\tilde{g} \tilde{h} = \tilde{g}$  by regularity and the fact that  $\mathfrak{M}$  is generated by the  $[u]$ .

Next, let  $\sigma_L : L \rightarrow \mathfrak{M}$  be the locale map for which the inverse image mapping  $\sigma_L^* : \mathfrak{M} \rightarrow L$  takes  $[u]$  to  $\bigvee k(u)$ . Then  $\sigma_L$  is natural in  $L$ : for any  $h : M \rightarrow L$ ,

$$h^* \sigma_L^*([u]) = h^*(\bigvee k(u)) = \bigvee \{ h^*(v) \mid v \ll u \},$$

while

$$\sigma_M^* \tilde{h}^*([u]) = \bigvee \{ w \in M \mid w \ll h^*(u'), u' \ll u \} \leq h^* \sigma_L^*([u])$$

and hence  $\sigma_L h = \tilde{h} \sigma_M$ . Moreover,  $\sigma_L$  is dense, a sublocale embedding iff  $L$  is completely regular, and an isomorphism iff  $L$  is, in addition, compact, by a familiar result of Vermeulen [9].

In all this proves

**PROPOSITION 1.** *For any locale  $L$ , the compact completely regular locale  $\mathfrak{M}$  is the Stone-Ćech compactification of  $L$ , with reflection map  $\sigma_L : L \rightarrow \mathfrak{M}$ .*

**REMARK.** From axiom (4) one readily derives, by induction, the entailment

$$u_1 \vee \cdots \vee a_n \in P \vdash u'_1 \in P \vee \cdots \vee u'_n \in P \text{ whenever } u_i \ll u'_i \text{ for all } i$$

which specialises to

$$\begin{aligned} \text{true} \vdash u'_1 \in P \vee \cdots \vee u'_n \in P \\ \text{whenever there exist } u_i \ll u'_i \text{ for all } i \text{ such that } u_1 \vee \cdots \vee u_n = e. \end{aligned}$$

We note that replacing (4) by the latter, for a completely regular locale  $L$ , produces the propositional theory of a *regular Cauchy filter* on  $L$  in the uniformity of  $L$  corresponding to the

Stone-Ćech compactification. Further, simple calculations show that, in general, the propositional theory obtained by this replacement is equivalent to the original one.

### 3. OTHER COMPACTIFICATIONS

Next we turn to the weaker form of the Stone-Ćech compactification, the *compact regular reflection*, and the aim is to show that this, too, can be described by a propositional theory presented directly in terms of the locale concerned. Indeed, the theory involved here will just be a simple variant of the one introduced above, obtained by replacing the relation  $\ll$  by a suitable alternative.

The key notion here is the following. Calling a binary relation  $R$  on any set *interpolative* if  $xRy$  implies  $xRzRy$  for some  $z$  (or:  $R \subseteq R \circ R$ , in terms of relational composition), it is clear that the union of any set of interpolative relations is again interpolative. In particular, then, any given relation contains a largest interpolative relation.

Now, for any locale  $L$ , let  $\triangleleft$  be the largest interpolative relation contained in the familiar *rather below* (or: well inside) relation  $<$  on  $L$  where  $u < v$  iff  $u^* \vee v = e$ . Note that  $\ll$  is contained in  $\triangleleft$ , and, if the Axiom of Countable Dependent Choice holds, the two relations coincide, but in general this is not the case. Nonetheless, as shown in Banaschewski-Pultr [5], the relation  $\triangleleft$  has all the properties listed earlier for  $\ll$  which entered into the proof of Proposition 1. As a result, we have the analogous conclusion concerning the propositional theory, for any locale  $L$ , which is given by the basic propositions  $u \in P$  for each  $u \in L$  together with the axioms

$$(1') \quad 0 \in P \vdash \text{false}, \text{true} \vdash e \in P;$$

$$(2') \quad u \in P \wedge v \in P \vdash u \wedge v \in P;$$

$$(3') \quad u \in P \vdash \bigvee \{ v \in P \mid v \triangleleft u \};$$

$$(4') \quad u \vee v \in P \vdash u' \in P \vee v' \in P \text{ whenever } u \triangleleft u' \text{ and } v \triangleleft v'.$$

The filters on  $L$  described by (3') are called *strongly regular*, and the almost prime strongly regular filters ((4') and (3')) are exactly the maximal strongly regular filters, by the proof of the Lemma.

Denote by  $\mathfrak{M}$  the Lindenbaum locale of this new propositional theory. Then, by the proof of Proposition 1,  $\mathfrak{M}$  is compact regular, though not necessarily completely regular: using the same notation as before,  $v \triangleleft u$  does imply  $[v] < [u]$ , but the features of  $\ll$  which allowed the conclusion that  $[v] \ll [u]$  are missing for  $\triangleleft$ . Further, we again have a locale map, say  $\tau_L : L \rightarrow \mathfrak{M}$ , such that  $\tau_L^*([u]) = \bigvee \{ v \in L \mid v \triangleleft u \}$  for any  $u \in L$ , which is natural in  $L$  and dense, a sublocale embedding iff  $L$  is *strongly regular* in the sense that  $u = \bigvee \{ v \in L \mid v \triangleleft u \}$  for all  $u \in L$ , and an isomorphism iff  $L$  is, furthermore, compact.

Noting in addition that, on any compact regular locale  $L$ ,  $<$  is interpolative as a result of its normality, so that  $\triangleleft$  is  $\triangleleft$  and  $L$  is strongly regular, we obtain

**PROPOSITION 2.** *For any locale  $L$ , the compact regular locale  $\mathfrak{M}$  is the compact regular reflection of  $L$ , with reflection map  $\tau_L : L \rightarrow \mathfrak{M}$ .*

The following observations illustrate the scope of the kind of propositional theory used here.

To begin with, the compact zero-dimensional reflection of any locale  $L$ , obtained in Banaschewski [1] as the locale of ideals of the Boolean algebra of complemented elements of  $L$ , can be described by the propositional theory obtained from our original one by replacing  $\ll$  in the last two axioms with the relation

$$v \leq w \leq u \text{ for some complemented } w \in L.$$

In more streamlined form, these axioms can then be expressed as

$$u \in P \vdash \forall \{v \in P \mid v \leq u \text{ complemented}\}$$

$$u \vee v \in P \vdash u' \in P \vee v' \in P \text{ for any } u' \geq u, v' \geq v \text{ with } u', v' \text{ complemented.}$$

In a similar vein, the construction of compactifications by means of normal bases presented in Robertson [8] has a natural description in the present setting. For this, recall from [8] that a *normal basis* of a regular locale  $L$  is a subset  $N$  of  $L$  which join-generates  $L$  and satisfies the following conditions:

$N$  is closed under finitary meets and  $0 \in N$ ;

for each  $u \in N$ ,  $u^* \in N$  and  $u^{**} = u$ ; and

for any  $u, v \in N$ , if  $u < v$  then there exists  $w \in N$  such that  $u < w < v$ .

It is clear that the relation  $<$  on  $N$  has all the properties noted originally for  $\ll$  and later on for  $\triangleleft$ . On the other hand,  $N$  is a lattice (in fact, a Boolean algebra) with join

$$u \sqcup v = (u \vee v)^{**} = (u^* \wedge v^*)^*$$

in terms of join  $\vee$  in  $L$ . Now, the proof of Proposition 2 only uses finitary operations and the pseudo-complementation (for which  $v^* < u^*$  whenever  $u < v$ ), and consequently the following propositional theory has a compact regular Lindenbaum algebra  $\mathfrak{R}$ : the basic propositions are  $u \in P$ , for each  $u \in N$ , and the axioms are obtained from the original ones with  $<$  now replacing  $\ll$ , noting that in axiom 3 the join has to be taken in  $N$ : namely,  $u \sqcup v \in P$ . Furthermore, this setting determines a locale map  $L \rightarrow \mathfrak{R}$  whose inverse image mapping takes the element  $[u] \in \mathfrak{R}$  (which corresponds to  $u \in P$ ) to  $u$ , and by hypothesis this is a dense sublocale embedding. Thus we have an alternative proof of the result of Robertson [8] that any regular locale with a normal basis can be embedded in a compact regular locale.

We note that the analogous result in [8] concerning a somewhat different kind of basis, called a Fan-Gottesman normal basis, can also be obtained with the present technique.

Finally, it seems worth pointing out that the type of propositional theory considered here provides a *universal method of describing compactifications* in the sense that any dense sublocale embedding  $f: L \rightarrow K$  of a locale  $L$  into a compact regular locale  $K$  results from an appropriate theory of our kind. Specifically, one shows that  $K$  is isomorphic to the Lindenbaum algebra  $\mathfrak{R}$  of the version of our theory in which the original  $\ll$  is replaced by the relation

$$u \leq f^*(u'), u' < v', f^*(v') \leq v \text{ for some } u', v' \in K,$$

and that the composite of  $f: L \rightarrow K$  with the isomorphism  $K \rightarrow \mathfrak{R}$  is then the natural map  $L \rightarrow \mathfrak{R}$  resulting in this setting whose inverse image mapping is induced by the model of the theory which takes the proposition  $u \in P$  to the element  $u$ .

#### 4. MAXIMALITY AND DE MORGAN'S LAW

We conclude this note with the promised proof of the earlier assertion that the maximal completely regular filters in the sense considered here are exactly the completely regular filters that are maximal in the partial order given by subset inclusion (to be called order-maximal) provided the De Morgan Law holds. Explicitly, these are the proper completely regular filters  $P$  such that, for any proper completely regular filter  $F$ ,  $F \supseteq P$  implies  $F = P$ .

( $\Rightarrow$ ) By an obvious and generally valid argument any maximal completely regular filter is order-maximal.

( $\Leftarrow$ ) By the Lemma, it will be enough to show that any order-maximal completely regular filter  $P$  is almost prime. For any  $u \in L$ , let  $[u]$  be the completely regular filter  $\{v \in L \mid u \ll v\}$  and  $P \vee [u]$  the (completely regular) filter generated by  $P$  and  $[u]$ . Then, by order-maximality,  $\neg(0 \in P \vee [u])$  implies  $[u] \leq P$ , and hence

$$\neg(u'' \in P) \text{ implies } \neg(0 \in P \vee [u])$$

for any  $u'' \in [u]$ . Further, using the same observation for any other  $v \in L$  and  $v'' \in [v]$ , we see that

$$\neg(u'' \in P) \text{ and } \neg(v'' \in P) \text{ implies } \neg(0 \in P \vee ([u] \wedge [v]))$$

by the properties of double negation and the distributivity of the lattice of all filters in  $L$ .

Now, given any  $u \vee v \in P$  and  $u \ll u', v \ll v'$ , choose  $u \ll u'' \ll u'$  and  $v \ll v'' \ll v'$  by interpolativity. Then  $[u] \wedge [v] \leq P$ , and, because  $P$  is proper, the premise of the above implication is false, proving that  $\neg(u'' \in P)$  or  $\neg(v'' \in P)$  by the De Morgan Law. Hence, the desired result will follow if we can show that  $\neg(w \in P)$  implies  $w' \in P$  whenever  $w \ll w'$ . To this end, consider

$$Q = \{x \in L \mid y \ll x \text{ for some } y \text{ such that } \neg(y \in P)\}.$$

Observe that  $Q$  is trivially an upset, a filter by the properties of double negation, and clearly completely regular because  $\ll$  is interpolative. Further,  $Q \geq P$  by complete regularity, and  $Q$  is proper since, by its construction,  $0 \in Q$  implies  $\neg(0 \in P)$ , contradicting the properness of  $P$ . Hence, by order-maximality, one has that  $Q = P$ , yielding that  $w \ll w'$  and  $\neg(w \in P)$  implies  $w' \in P$ , as desired.

Note that the same arguments apply in all the other situations considered above.

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