Exercise Sheet 4

(*) questions contribute, Deadline 3pm 4th December 2006

1. **(*)** Let the CG iteration be applied to a positive definite matrix problem

$$A\mathbf{x} = \mathbf{b}.$$

While $\mathbf{r}_{k-1} \neq \mathbf{0}$, show that the algorithm proceeds without divisions by zero and we have

$$\mathcal{K}_k := < \mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{k-1} > = < \mathbf{b}, A\mathbf{b}, ..., A^{k-1}\mathbf{b} > .$$

2. For a postive definite matrix $A \in \mathbb{R}^{n \times n}$ show that $F_A(\mathbf{x}) := \sqrt{\mathbf{x}^T A \mathbf{x}}$ is a vector norm on $\mathbb{R}^n$.

3. Apply two iterations of the conjugate gradient method for the problem

$$A\mathbf{x} = \mathbf{b}, \text{ where } A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

to obtain $\mathbf{x}_2$. Calculate the residual at this iteration and comment on your answer.

4. For a given $A \in \mathbb{R}^{n \times n}$ and $\mu \in \mathbb{R}$, we may write the inverse iteration in the form: Given $\mathbf{x}^0 \in \mathbb{C}^n$
for $k = 1, 2, \ldots$
Solve

$$(A - \mu I)\mathbf{w} = \mathbf{x}^{k-1}$$
$$\mathbf{x}^k = \mathbf{w} / \|\mathbf{w}\|$$
$$\lambda^k = (\mathbf{x}^k)^* A \mathbf{x}^k$$

Assume A is complete and $\mathbf{x}^0 = \sum_{i=1}^n \alpha_i \mathbf{z}_i$, where $\{\mathbf{z}_i\}$ are the eigenvectors of A with corresponding eigenvalues $\{\lambda_i\}$. Under certain conditions, that are to be stated,

(a) Describe the eigenvalues and eigenvectors of $(A - \mu I)^{-1}$ in terms of μ and the eigenvalues, eigenvectors of A.

(b) Show that

$$\mathbf{x}^r = \frac{1}{K_0 K_1 K_2 \cdots K_{r-1}} \sum_{i=1}^n \alpha_i \left(\frac{1}{\lambda_i - \mu} \right) \mathbf{z}_i,$$

where $K_j = \|(A - \mu I)^{-1} \mathbf{x}^j\|$.

(c) For any $1 \leq J \leq n$, show that

$$\|K_0 K_1 K_2 \cdots K_{r-1} (\lambda_J - \mu)^r \mathbf{x}^r - \alpha_J \mathbf{z}_j\| \leq \left| \frac{\lambda_J - \mu}{\lambda_I - \mu} \right|^r \sum_{i=1, i \neq J}^n |\alpha_i| \, \|\mathbf{z}_i\| ,$$

where the properies of the index I are to be stated.

(d) Describe how a suitable choice of μ may produce a fast convergence of $\mathbf{x}^r$ to an element in the subspace spanned by the eigenvector $\mathbf{z}_J$.

5. Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Apply three iterations of the power method to A to find the largest eigenvalue.

(b) Apply three iterations of the inverse iteration, with $\mu = 1.3$ to find the smallest eigenvalue of A.

(c) Apply three iterations of the Rayleigh quotient iteration, with a suitable initial μ to find the middle eigenvalue of A.

6. **(*) Computation/MATLAB**

(a) By setting the random number generator seed to 20 construct a 200 by 200 random matrix, B. Setting the seed to 50 construct a column vector $b200$ of length 200.

(b) Now construct

$$A200 = I + B^t B.$$

Calculate the largest and smallest eigenvalues in magnitude of $A200$.

(c) Construct the Conjugate Gradient function given in the lecture notes. Setting $tol = 10^{-6}$ find the number of iterations required for the conjugate gradient method to obtain this tolerance.

(d) Repeat the previous three steps but with a 400 by 400 matrix, $A400$, and then an 800 by 800 matrix, $A800$. Comment on the results.