A Finite Element Method for the Miscible Displacement Problem
Convergence of a discontinuous Galerkin / Raviart-Thomas Method

Sören Bartels**, Max Jensen*, Rüdiger Müller***

* Department of Mathematical Sciences
  University of Durham, UK

** Institut für Numerische Simulation
  Universität Bonn, Germany

*** Institut für Mathematik
  Humboldt-Universität zu Berlin, Germany

Supported by the DAAD/DST PPP programme and Matheon.

Brighton, 15. November 2007
Outline

Introduction to Miscible Oil Recovery

Equations of Incompressible Miscible Displacement

The Finite Element Method

Convergence Analysis
  Existence of Numerical Solutions
  Stability
  Compactness
  Convergence
  Intermission: Gradients of dG Functions
  Weak Solution
  Trace

Numerical Experiment
The Classical Setting

Primary Recovery

- pressure in oil reservoir higher than hydrostatic pressure
  - reservoir fluids driven to well.
- on average 10% of the oil can be recovered

Secondary Recovery

- water is injected to maintain pressure and to push the oil forward.
- between 15 - 60 % of the oil can be recovered
The low recovery rates can be explained by

- Low sweep efficiency at the macroscopic scale:
  \( \text{viscosity of water} \ll \text{viscosity of oil} \)
  \( \Rightarrow \) fingering

- Low displacement efficiency at the pore scale:
  capillary forces trap the oil

Tertiary Recovery

- chemical methods
  (e.g. alkaline, surfactant, polymer flooding)
- thermal methods
  (steam drive)
- miscible methods
  (e.g. \( \text{CO}_2 \) and hydrocarbon flooding)
Economically the most relevant tertiary method is miscible displacement by CO$_2$.

In 2004 about 4% of the US national total was produced with CO$_2$ recovery.

Source: Oil & Gas Journal biennial EOR Survey 2004
Equations of Incompressible Miscible Displacement

Unkowns:

- concentration $c$
- pressure $p$
- Darcy velocity $u$

Darcy’s Law:

$$u = -\frac{K}{\mu(c)}(\nabla p - \rho g)$$

Incompressibility:

$$\text{div} \, u = q^I - q^P$$

Concentration Equation:

$$\phi \frac{\partial}{\partial t} c - \text{div} \, (\mathbb{D}(u) \nabla c) + u \cdot \nabla c + q^I c = \hat{c} q^I$$

Parameter:

- porosity $\phi$
- viscosity $\mu$
- gravity $g$
- injection source $q^I$
- injected concentration $\hat{c}$
- absolute permeability $K$
- density $\rho$
- diffusion-dispersion coefficient $\mathbb{D}$
- production sink $q^P$
Equations of Incompressible Miscible Displacement

**Unknowns:**
- concentration $c$
- pressure $p$
- Darcy velocity $u$

**Darcy’s Law:** $\forall \nu, q$

\[
(\mu(c) K^{-1} u, \nu) - (p, \text{div} \nu) = (\rho(c) g, \nu)
\]

**Incompressibility:** $\forall q$

\[
(q, \text{div} u) = (q^I - q^P, q)
\]

**Concentration Equation:** $\forall w$

\[
\int_0^T - (\phi c, \partial_t w) + (D(u) \nabla c, \nabla w) + (u \cdot \nabla c, w) + (q^I c, w) - (\hat{c} q^I, w) dt = 0
\]

**Parameter:**
- porosity $\phi$
- viscosity $\mu$
- gravity $g$
- injection source $q^I$
- injected concentration $\hat{c}$
- absolute permeability $K$
- density $\rho$
- diffusion-dispersion coefficient $D$
- production sink $q^P$
Equations of Incompressible Miscible Displacement

**Unknowns:**
- concentration $c$
- pressure $p$
- Darcy velocity $u$

**Darcy’s Law:** $\forall v, q$

$$
(\mu(c)K^{-1}u, v) - (p, \text{div } v) = (\rho(c)g, v)
$$

**Incompressibility:** $\forall q$

$$(q, \text{div } u) = (q^I - q^P, q)$$

**Concentration Equation:** $\forall w$

$$
\int_0^T - (\phi c, \partial_t w) + (\mathbf{D}(u) \nabla c, \nabla w) + (u \cdot \nabla c, w) + (q^I c, w) - (\hat{c} q^I, w) dt = 0
$$

**Regularity:**
- $K, \phi, q^I, q^P$ belong to $L^\infty$.
- The domain $\Omega$ is Lipschitz regular.

**growth condition**

$$(1 + |u|) |\xi|^2 \lesssim \xi^T \mathbf{D}(u, x) \xi \lesssim (1 + |u|) |\xi|^2$$
Initial and Boundary Conditions

Initial Conditions: at $t = 0$

\[ c(0, \cdot) = c_0. \]

Boundary Conditions: on $(0, T) \times \partial \Omega$

\[ u \cdot n = 0, \quad (\mathbb{D}(u) \nabla c) \cdot n = 0. \]

Remark

*Neumann boundary conditions are most relevant in practice.*

*The analysis can directly be carried over to Dirichlet and mixed boundary conditions.*
Notation and Basic Assumptions

- shape-regular triangulation $\mathcal{T}$
- skeleton $\mathcal{E}$
- elements denoted by $T$, edges by $E$ or similar
- $S_c = S_c(\mathcal{T})$ elementwise polynomials of degree $k$
- $S_u := \mathcal{RT}_N^k$ Raviart-Thomas space (with Neumann boundary conditions)
- $S_p$ elementwise polynomials of degree $k$
- average $\{v\} := (v^- + v^+)/2.$
Definition of the RT / DG scheme

Fix $c^0_h \in S_c$.

For $1 \leq j \leq M$ find $(u_h^j, p_h^j, c_h^j) \in S_u \times S_p \times S_c$ such that

- for all $(v, q) \in S_u \times S_p$
  
  \[
  \left( \mu (c_h^j) K^{-1} u_h^j, v \right) - (p_h^j, \text{div} \, v) = (\rho g, v),
  \]
  
  \[
  (q, \text{div} \, u_h^j) = (q^I - q^p, q),
  \]

- for all $w \in S_c$
  
  \[
  \left( \phi d_t c_h^j, w \right) + B_d (c_h^j, w; u_h^j) + B_{cq} (c_h^j, w; u_h^j) = (\hat{c}_h q^I, w)
  \]

\[
d_t \quad \text{implicit Euler operator}
\]
\[
B_d \quad \text{discretisation of diffusion term}
\]
\[
B_{cq} \quad \text{discretisation of convection, injection and production terms}
\]
The dG scheme: Diffusion

\( \mathbb{D} \) may have little more than \( L^2 \)-regularity. Hence it is replaced by an elementwise polynomial \( \mathbb{D}_h \) such that

\[
\| \mathbb{D}_h(u_h) - \mathbb{D}(u_h) \|_T \lesssim h_T (1 + \|u_h\|_T), \quad T \in \mathcal{T}.
\]

Then set

\[
B_d(c_h, w_h; u_h) := (\mathbb{D}_h(u_h) \nabla_h c_h, \nabla_h w_h) - (n_{\mathcal{E}} [c_h], \{\mathbb{D}_h(u_h) \nabla_h w_h\})_{\mathcal{E}_j} \Omega
\]
\[
- (n_{\mathcal{E}} [w_h], \{\mathbb{D}_h(u_h) \nabla_h c_h\})_{\mathcal{E}_j} \Omega + (\sigma^2[c_h], [w_h])_{\mathcal{E}_j} \Omega
\]

where

\[
\sigma^2 : \mathcal{E}_j \Omega \to \mathbb{R}, \quad x \mapsto C_\sigma \max \{ n_{\mathcal{E}}^T \mathbb{D}_h(u^+_h, x) n_{\mathcal{E}}, \ n_{\mathcal{E}}^T \mathbb{D}_h(u^-_h, x) n_{\mathcal{E}} \} / h_{\mathcal{E}}.
\]
Convection, injection and production terms: ... the first try ...

\[ B_{cq}(c_h, w_h; u_h) := (u_h \nabla_h c_h, w_h) + (q^I c_h, w_h) - \sum_{T \in T_j} ((u_h \cdot n_T) - [c_h], w_h^+)_{\partial T \setminus \partial \Omega} \]

Problem: To establish semi-definiteness one requires control on \( \text{div}_h u_h \).
Convection, injection and production terms: \( \text{second try} \)

\[
B_{cq}(c_h, w_h; u_h) := \frac{1}{2} \left( (u_h \nabla_h c_h, w_h) - (u_h c_h, \nabla_h w_h) + ((q^I + q^P)c_h, w_h) \right.
+ \sum_{T \in T^j} \left[ \frac{c_h^+}{(u_h \cdot n_T) + [w_h]} \right]_{\partial T \setminus \partial \Omega} - \left[ (u_h \cdot n_T - [c_h], w_h^+) \right]_{\partial T \setminus \partial \Omega}
\]

Solution: We used that

\[
u \cdot \nabla c = \frac{1}{2} u \cdot \nabla c + \frac{1}{2} \text{div} (uc) - \frac{1}{2} (q^I - q^P) c,
\]

before discretising the concentration equation.

Now semi-definiteness of \( B_{cq} \) follows independent of properties of \( u_h \).

In particular, no ‘cut-off’ of \( u_h \) necessary!
▶ Feng 1994:
  ▶ existence of weak solutions
  ▶ uniqueness of semi-classical solutions

▶ Chen, Ewing 1999:
  ▶ generalisation of existence proof (BCs, domain, gravity)
  ▶ existence by Brouwer’s thm applied to discretisation of a regularised BVP.

▶ Sun, Revière, Wheeler 2002:
  ▶ a priori bound for initially presented RK / DG method, assuming unique solution

\[(u, c) \in C_B \cap L^\infty(0, T; H^{l+1/2}) \times C_B \cap L^\infty(0, T; H^m)\]

and \(\partial_t c \in L^2(0, T; H^n)\).
The main question in the analysis of this PDE are

1. Control on $\text{div}_h u_h$.
2. Weak conditions on hanging nodes. (not shown here)
3. Deal with changing weights.
4. Low regularity of coefficients.
5. Convergence of the dG gradient.

Outline

existence of numerical solutions 1

$\leadsto$ stability 1, 2

$\mathcal{I}$ compactness 3

$\leadsto$ convergence

$\leadsto$ \{ differential equation 4, 5

$\leadsto$ initial condition 3
Existence of Numerical Solutions

- Consider the residual $\mathcal{R}(c_h)[w_h]$, defined by
  $$ k_j^{-1}(\phi(c_h - c_h^{-1}), w_h) + B_d(c_h, w_h; u_h) + B_{cq}(c_h, w_h; u_h) - (\hat{c} q^I, w_h). $$

- Using the half-primal dG method for the lower-order terms, one finds
  $$ B_d(c_h, c_h; u_h) + B_{cq}(c_h, c_h; u_h) \gtrsim (D_h(u_h) \nabla_h c_h, \nabla_h c_h) + ((q^I + q_P) c_h, c_h) + (\sigma^2 [c_h], [c_h])_\varepsilon. $$

  independent of the properties of $u_h$.

- One concludes that $\mathcal{R}(c_h)[c_h]$ is non-negative for sufficiently large $c_h$.

- Using a corollary of Brouwer’s fixed point theorem shows that $\Phi$ has a root $c^j_h$.

Theorem (Unconditional Existence of Numerical Solutions)

In each time step the numerical scheme has a solution.
The Raviart-Thomas FEM satisfies
\[ \| u_h^j \| + \| \text{div} u_h^j \| + \| p_h^j \| \lesssim \| g \| + \| q^l - q^p \|. \]

Equally we have with
\[ \| \nabla_h v \|^2_T := \| v \|^2 + \| \nabla_h v \|^2 + \| h^{-1/2}_E [v] \|^2_E \]
that
\[ \| \phi^{1/2} c_h^j \|^2 + \int_0^{t_j} k \| \phi^{1/2} d_t c_h \|^2 + \| c_h^j \|^2_T \, dt \leq \text{`data'} \]
for all \( j \) by expanding the backward Euler difference into a telescope sum.

We assume \( \| v \|_{L^4(\Omega)} \lesssim \| v \|_T \).

A little more elaborate is the proof of
\[ \| \phi d_t c_h \|_{L^2(0,T;H^1(\Omega)^*)} \leq C, \]

independently of the mesh size, time step and polynomial degree.
The space $L^2(0, T, B)$ is a space of functions

$$f : (0, T) \rightarrow B, \ t \mapsto f(t) \in B.$$ 

E.g. if $B = H^1(\Omega)$ then $f$ attains at every time a spatial function in $H^1(\Omega)$. 

Bochner Spaces
Compactness

**Theorem (Aubin-Lions)**

Consider the reflexive Banach spaces

\[ B_0 \xrightarrow{compact} B \hookrightarrow B_1. \]

Then the space

\[ W(B_0, B_1) := \{ w \in L^2(0, T, B_0) : \partial_t w \in L^2(0, T, B_1) \} \]

is compactly embedded in \( L^2(0, T; B) \).

The previous stability proof leads to the choice \( B_1 = H^\ell(\Omega)^* \).

However, how to choose \( B_0 \)?
Ideally we would find a space $\mathcal{S}$ which has the following properties:

1. $S_c(T)$ embeds continuously into $\mathcal{S}$ with $\|v\|_T \gtrsim \|v\|_{\mathcal{S}}$.
2. $\mathcal{S}$ embeds compactly into $L^2(\Omega)$.
3. $\mathcal{S}$ is a reflexive space.

Then $\mathcal{S}$ is a suitable choice for $B_0$. 
Ideally we would find a space $\mathcal{I}$ which has the following properties:

(1) $S_{c}(T)$ embeds continuously into $\mathcal{I}$ with $\|v\|_{T} \gtrsim \|v\|_{\mathcal{I}}$.

(2) $\mathcal{I}$ embeds compactly into $L^{2}(\Omega)$.

(3) $\mathcal{I}$ is a reflexive space.

Then $\mathcal{I}$ is a suitable choice for $B_{0}$.

We know:

- $S_{c}(T)$ embeds continuously into $BV(\Omega)$ with $\|v\|_{T} \gtrsim \|v\|_{BV(\Omega)}$.
- $S_{c}(T)$ embeds continuously into $L^{4}(\Omega)$ with $\|v\|_{T} \gtrsim \|v\|_{L^{4}(\Omega)}$.
- $BV(\Omega)$ embeds compactly into $L^{1}(\Omega)$ but is not reflexive.
- $L^{4}(\Omega)$ is a reflexive space but does not embed compactly into a useful space.
Can we interpolate $BV(\Omega)$ and $L^4(\Omega)$ to obtain a reflexive space which embeds compactly?

**Example (Well-known Interpolation Spaces)**

- $L^p(\Omega)$, $1 < p < \infty$, gained by interpolating $L^1(\Omega)$ and $L^\infty(\Omega)$.
- Fractional Sobolev spaces by interpolating Sobolev spaces of integer order.
- Hölder continuous functions by interpolating Lipschitz continuous functions with $C(\Omega)$. 
Can we interpolate $BV(\Omega)$ and $L^4(\Omega)$ to obtain a reflexive space which embeds compactly?

First try:
Can we interpolate $BV(\Omega)$ and $L^4(\Omega)$ to obtain a reflexive space which embeds compactly?

First try:

$$
L^1(\Omega) \quad \quad \quad \quad [L^1(\Omega), L^4(\Omega)]_{1/3} \quad \quad \quad \quad L^4(\Omega)
$$

$\uparrow$

$BV(\Omega)$

compact

$\downarrow L^4(\Omega)$
Can we interpolate $BV(\Omega)$ and $L^4(\Omega)$ to obtain a reflexive space which embeds compactly?

First try: Reflexivity $\checkmark$, Compactness $??$

It is a longstanding open question whether complex interpolation respects compactness and only partial results exists in the current literature.
Luckily, we can apply a result by M. Cwikel and N.J. Kalton (1995): If

\[ Z \quad \alpha \quad \uparrow \quad \text{compact} \quad \uparrow \quad \theta \quad \text{compact} \quad Y_1 \]

\[ X_0 \quad \beta \quad \uparrow \quad \theta \quad \uparrow \quad X_1 \]

then

\[ Z \quad \alpha \quad \uparrow \quad \text{compact} \quad \uparrow \quad \theta \quad \text{compact} \quad Y_1 \]

\[ X_0 \quad \beta \quad \uparrow \quad \theta \quad \uparrow \quad X_1 \]
Luckily, we can apply an result by M. Cwikel and N.J. Kalton (1995): If

\[ L^1(\Omega) \xrightarrow{\text{compact}} L^{4/3}(\Omega) \xrightarrow{} L^2(\Omega) \xrightarrow{} L^4(\Omega) \]

\[ \text{BV}(\Omega) \cap L^4(\Omega) \xrightarrow{} L^4(\Omega) \]

then

\[ Z \xrightarrow{\text{compact}} Y_0 = [Z, Y_1]_\alpha \xrightarrow{} [Y_0, Y_1]_\theta \xrightarrow{\text{compact}} Y_1 \]

\[ X_0 \xrightarrow{} [X_0, X_1]_\theta \xrightarrow{} X_1 \]
Compactness and $\mathcal{S}$

Luckily, we can apply an result by M. Cwikel and N.J. Kalton (1995): If

$$L^1(\Omega) \quad \longrightarrow \quad L^{4/3}(\Omega) \quad \longrightarrow \quad L^2(\Omega) \quad \longrightarrow \quad L^4(\Omega)$$

then

$$BV(\Omega) \cap L^4(\Omega) \quad \longrightarrow \quad [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2} \quad \longrightarrow \quad L^4(\Omega)$$

The space $\mathcal{S} := [BV(\Omega) \cap L^4(\Omega), L^4(\Omega)]_{1/2}$ is also reflexive.
Convergence of the Concentration

Hence we can apply the Aubin-Lions lemma to

$$W(\mathcal{I}, H^\ell(\Omega)^*) = \{ v \in \mathcal{I} : \partial_t v \in H^\ell(\Omega)^* \}.$$ 

**Attention!**

If you were lost with the interpolation spaces, come back now!

**Theorem**

Let $$(u_h, p_h, c_h)_h$$ be a sequence of numerical solutions and $$h, k \to 0$$. Then there exists

$$c \in W(\mathcal{I}, H^\ell(\Omega)^*)$$

such that, possibly after passing to a subsequence,

$$c_h \to c \quad \text{in } L^2(\Omega T), \quad d_t c_h \rightharpoonup \partial_t c \quad \text{in } L^2(0, T; H^\ell(\Omega)^*).$$
The concentration $c_h$ acts as a perturbation on the mixed system. This allows to apply Strang’s Lemma.

**Theorem**

Let $(u_h, p_h, c_h)_h$ be a sequence of numerical solutions and $c_h \to c$ in $L^2(\Omega_T)$ as $h, k \to 0$.

Then there exists an unique pair of functions

$$u \in L^\infty(0, T; H_N(\text{div}; \Omega)) \text{ and } p \in L^\infty(0, T; L^2_0(\Omega))$$

such that, possibly after passing to a subsequence,

$$u_h \to u \quad \text{in } L^\infty(0, T; H_N(\text{div}; \Omega)), \quad p_h \to p \quad \text{in } L^\infty(0, T; L^2_0(\Omega))$$

as $h, k \to 0$.

Furthermore, $(u, p, c)$ solve Darcy’s law and the incompressibility equation.
Intermission: Gradients of dG Functions

Example

The following sequence is $\| \cdot \|_{\mathcal{T}}$-bounded:

$$v_i : (0, 1) \rightarrow \mathbb{R}, \ x \mapsto x - \max\{\frac{n}{i} \in \mathbb{R} : n \in \mathbb{N}, \frac{n}{i} \leq x\}.$$  

The functions $v_3 (\cdots)$, $v_6 (---)$ and $v_{12} (\ldots)$. Then

- $v_i \rightarrow 0$ in $L^2(0, 1)$,
- $\nabla_h v_i \equiv 1 \rightarrow 1$ in $L^2(0, 1)$, (abs. cont. component),
- $\sum_E ([v_i] \cdot n_E, \cdot)_E \rightarrow -1$ in $H^{-1}(0, 1)$, (jump component),
- $\nabla v_i \rightarrow 0$ in $H^{-1}(0, 1)$, (distributional gradient).
The previous example shows that the gradient of the limit is composed of the jump parts and the absolutely continuous parts $\nabla_h c_h$.

Hence it is insufficient to consider $\nabla_h c_h$ alone.

**Example**

In 2D the limit $\lim \nabla_h c_h$ does not need to be a gradient anymore.
Intermission: Gradients of dG Functions

In general, $\| \cdot \|_T$-bounded sequences have smooth cluster points.

**Theorem**

Consider a bounded sequence $(v_i)_i$ of dG functions:

$$\| v_i \|_T < C_*$$

Let $h \to 0$. Then there is a $v \in H^1(\Omega)$ with

$$\| v \|_{H^1(\Omega)} \lesssim C_*$$

such that, after passing to a subsequence,

$$v_i \to v \quad \text{in } L^2(\Omega),$$
$$\nabla v_i \to \nabla v \quad \text{in } H^{-1}(\Omega).$$

Hence we know $\nabla c \in H^1(\Omega)$. 
The Limit is a Weak Solution

We want to show that for all smooth $\nu$ and their discrete approximation $\nu_h$

$$
(\mathbb{D}(u) \nabla c, \nabla \nu) = \lim B_d(c_h, \nu_h; u_h),
$$

$$
(u \cdot \nabla c, \nu) + (q^l c, \nu) = \lim B_{cq}(c_h, \nu_h; u_h),
$$

$$
(\phi c, \partial_t \nu) = -\lim (\phi d_t c_h, \nu_h).
$$

Identity (1) is the hardest and we shall focus on it.

Clearly, to examine (1), one needs to consider absolutely continuous and jump part of $c$’s derivatives — however, now in combination with the nonlinear, possibly singular weight $\mathbb{D}(u)$.

A combination of integration-by-parts and $c_h \rightarrow c$ in $L^2(\Omega)$ shows that

$$
(\mathbb{D}(u) \nabla c, \nabla \nu) = \lim B_d(c_h, \nu_h; u_h) + (c - \{c_h\}, n_\mathcal{E} \cdot [\mathbb{D}(u_h) \nabla_h \nu_h])_{\mathcal{E}}.
$$
The Limit is a Weak Solution

Hence the critical term is \((c - \{c_h\}, n_E \cdot [\mathbb{D}_h(u_h) \nabla h v_h])_E\).

Inverse inequalities show that \(\sum_T \|[\mathbb{D}_h(u_h) \nabla h v_h]/h_T^{1/2}\|^2_{\partial T}\) is bounded.

Therefore the term

\[
(c - \{c_h\}, n_E \cdot [\mathbb{D}_h(u_h) \nabla h v_h])_E
\]

\[
\lesssim \sum_{T \in T} \|h_T^{1/2} (c - c_h^+)\|_{\partial T} \|[\mathbb{D}_h(u_h) \nabla h v_h]/h_T^{1/2}\|_{\partial T}
\]

\[
\lesssim \sum_{T \in T} \|c - c_h\|_{L^2(T)}^2 + h_T \|c - c_h\|_{L^2(T)} (c - c_h)_{H^1(T)}
\]

converges to 0 as \(h \to 0\).

Theorem

The pair \((u, c)\) satisfies the concentration equation.
To see that the initial conditions are satisfied we need a trace operator. Let us first look at the standard treatment of initial conditions.

**Theorem (Lions)**

*Consider the Hilbert spaces*

\[ H_0 \hookrightarrow H \hookrightarrow H_1. \]

*Then*

\[ W(H_0, H_1) \hookrightarrow C_b(0, T; [H_0, H_1]^{1/2}). \]
We were not able to use this theorem directly because the problem appeared to us as follows:

To get values \( L^2(\Omega) = [H_0, H_1]^{1/2} \) we were inclined to choose \( H = L^2(\Omega) \) and \( H_1 = H_0' \).

To keep all solutions:

\[
H_0 \supset W_{u(t)} := \{ \nu \in H^1(\Omega) : \| \mathcal{D}(u(t))^{1/2} \nabla \nu \| < \infty \}.
\]

Out best stability bound on \( \partial_t c \) demands:

\[
H^1 \supset W_{u(t)}^* \implies H_0 \subset W_{u(t)}.
\]

But choosing \( H_0 = H_0(t) = W_{u(t)} \) has the disadvantages:

- Different spatial function spaces at different times (\( \leadsto \) not a Bochner space).
- Smooth functions may not be dense in \( W_{u(t)} \) (e.g. is a Muckenhoupt condition satisfied?).
Trace

The theorem by Lions shows that

\[ W(\mathcal{S}, H^\ell(\Omega)^*) \hookrightarrow C_b(0, T; [\mathcal{S}, H^\ell(\Omega)^*]_{1/2}^*) \hookrightarrow C_b(0, T; H^\ell(\Omega)^*) \].

Because of the stability bound, piecewise linear interpolations \( \tilde{c}_h \) of \( c_h \) are bounded in \( C_b(0, T; L^2) \).

Due to Mazur’s theorem finite convex combinations \( \bar{c}_h \) of \( \tilde{c}_h \) converge strongly in \( C_b(0, T; H^\ell(\Omega)^*) \).

As \( H^\ell(\Omega) \) is a strongly dense subset of \( L^2(\Omega) \) and

\[ \lim \langle v, \bar{c}_h(t) \rangle \to \langle v, c(t) \rangle, \quad \forall v \in H^\ell(\Omega), t \in [0, T], \]

one finds \( \bar{c}_h(t) \rightharpoonup c(t) \) in \( L^2(\Omega) \).

Pointwise weak convergence and boundedness give weak convergence in \( C(0, T; L^2(\Omega)) \), hence \( \tilde{c}_h \rightharpoonup c \).

Finally, the initial conditions are satisfied by continuity of the trace in \( C(0, T; L^2(\Omega)) \).
Example: Flow on an L-shape Domain

concentration at $t = 3$

$h = 1/16, k = 1/4$

$h = 1/32, k = 1/8$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>molecular diffusion</td>
<td>$10^{-6}$</td>
<td>mobility ratio</td>
<td>41</td>
</tr>
<tr>
<td>longitudinal dispersivity</td>
<td>$1.8 \times 10^{-4}$</td>
<td>permeability</td>
<td>0.01</td>
</tr>
<tr>
<td>transverse dispersivity</td>
<td>$1.8 \times 10^{-4}$</td>
<td>porosity</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Extensions in forthcoming paper:

- The polynomial degree $\ell$ is not necessarily uniform, but $\geq 1$ and bounded.
- The meshes $\mathcal{T}_u$ for $u$ and $p$ coincide.
- Treatment of hanging nodes.
- Meshes for $c$ may be a refinement of $\mathcal{T}_u$.
- The meshes may change in time, subject to a compatibility assumption

$$\tau := \sup \left( \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \frac{1}{k_j} \| P_{j-1} (\text{Id} - P_j) w(t, \cdot) \|^2 \, dt \right)^{1/2}$$

subject to $\|w\|_{L^2(0,T; H^{p_0}(\Omega))} \leq 1$. We remark that

$$\tau \lesssim \sup_j \sup \left\{ h_{\mathcal{T}}^{2\ell}/k_j : T \in \mathcal{T}_c^j, T \not\in \mathcal{T}_c^{j-1} \right\}.$$
Extensions in forthcoming paper:

- The polynomial degree $\ell$ is not necessarily uniform, but $\geq 1$ and bounded.
- The meshes $\mathcal{T}_u$ for $u$ and $p$ coincide.
- Treatment of hanging nodes.
- Meshes for $c$ may be a refinement of $\mathcal{T}_u$.
- The meshes may change in time, subject to a compatibility assumption

$$
\tau := \sup_{w} \left( \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \frac{1}{k_j} \| P_{j-1}(\text{Id} - P_j) w(t, \cdot) \|^2 \mathrm{d}t \right)^{1/2}
$$

subject to $\| w \|_{L^2(0,T; H^{p_0}(\Omega))} \leq 1$. We remark that

$$
\tau \lesssim \sup_j \sup \left\{ \frac{h^{2\ell}_T}{k_j} : T \in \mathcal{T}_c^j, T \notin \mathcal{T}_c^{j-1} \right\}.
$$

Thank you for your attention!
Bibliography


Assumptions in the Presentation

- The polynomial degree $\ell$ is uniform and $\geq 1$.
- $\|v\|_{L^4(\Omega)} \lesssim \|v\|_{T^j}$. 
- The meshes for $u$, $p$ and $c$ coincide.
- The meshes do not change in time.
- $h_{\mathcal{E}(T^j_c)} \lesssim h_{T^j_c}$ on the restriction to $\mathcal{E}(T^j_c)$.

Definition of $h_T$ and $h_E$. 

![Diagram showing the definition of $h_T$ and $h_E$.]
Consider an L-shaped domain.

Selecting suitable source terms one can find solutions \((u_1, p_1, c_1)\) and \((u_2, p_2, c_2)\) such that at the re-entrant corner:

- \(u_1\) has a strong pole, \(c_1\) has a mild pole.
- \(c_2\) has a strong pole, \(u_2\) has a mild pole.
- \((\mathbb{D}(u_1) \nabla c_1, \nabla c_2)\) diverges.

Hence one cannot apply standard arguments of the theory of monotone operators, including those concerning the convergence of Galerkin methods.

Nevertheless under the \textit{hypothesis} that \(u\) is bounded, one can show pseudo-monotone behaviour of \((\mathbb{D}(u) \nabla c, \nabla w)\).

This explains why our analysis differs in that point from [Sun, Revière, Wheeler 2002], who assume \(u \in C_b(0, T; \Omega) \cap L^\infty(0, T; H^{1/2}(\Omega))\).
Intermediate Time (Previous Example)
High Resolution (Previous Example)
Miscible Displacement Flow

Source: C.M. Marle, Critical Reports on Applied Chemistry, vol. 33
Miscible Displacement

Source: C.M. Marle, Critical Reports on Applied Chemistry, vol. 33
Compressibility of CO$_2$