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SUMMARY. This note summarises three intertwining observations, concerned with non-commutative C*-algebras, with constructive foundations for quantum mechanics, and with non-commutative logics, made at the Category Meeting at Oberwolfach in September, 1983 [6] .

1.

The extension of the Gelfand-Naimark representation to non-commutative C*-algebras introduced by Giles and Kummer [4] (and independently, from a different viewpoint, by Akemann [1]) leads one to consider, for any C*-algebra A , the lattices of closed left ideals

$$\mathrm{Lid}\, B \leftrightarrows \mathrm{Lid}\, A$$

of the C*-algebra A , and of an atomic W*-algebra B in which A is canonically embedded, together with mappings which extend and specialise closed left ideals. These lattices may be considered to be the non-commutative generalisation of those of open subsets and of arbitrary subsets, respectively, of the maximal spectrum of a commutative C*-algebra. Explicitly, B is determined by the irreducible representations of A , giving a W*-algebra in which lie the projections belonging to the spectral representation

$$a = \int \lambda\, dE_\lambda$$

of any self-adjoint element $a \in A$. Moreover, the lattice of projections of B is isomorphic to that of its closed left ideals.

The lattice of closed left ideals of A is embedded in that of closed left ideals of B , yielding those projections that are *open* in the sense that each arises in the form

$$a^{-1}(r,s) = \int_{(r,s)} dE_\lambda\ ,$$

for the spectral representation of some self-adjoint $a \in A$ and some open interval (r,s) . Then the Gelfand-Naimark representation theorem states that the elements $a \in B$ which belong to the C*-algebra A are exactly those for which the inverse image of each open interval (r,s) , in the above sense, is open: that is, those that are *continuous*. The representation is exactly the Gelfand representation in the case of a commutative C*-algebra.

It may be asked to what extent this representation of a C*-algebra yields a non-commutative extension of Gelfand duality, with this pair of lattices being the non-commutative analogue of the maximal spectrum of a commutative C*-algebra. In particular, there is a question of characterising the lattices which appear in this way, and of whether a maximal spectrum of this kind actually determines the C*-algebra isometrically. One conclusion is immediately evident, namely, that the lattices involved are not distributive. Moreover, the embedding of the lattice $\mathrm{Lid}\, A$ in the lattice $\mathrm{Lid}\, B$ appears to be necessary to the description of the maximal spectrum in this sense, at least at first sight.

2.

In considering constructive foundations of physics, in lectures at Oxford in 1975 [5] , I proposed the concept of propositional axiomatisations of physical theories, analogous to those appearing in defining constructive spectra within mathematics. The fundamental characteristic required was that any description of the physical world ought to be phrased in terms which involve minimal preconceptions concerning its nature. The aim was to seek ways of obtaining constructive foundations for quantum mechanics, avoiding, in particular, any initial introduction of Hilbert space. The idea was that, in examining the physical world, one is naturally led to introduce propositions of the form

$$a \in (r,s)$$

in which r,s denote rational numbers, representing an assertion that the observable a was found to yield a value within the interval (r,s) . The concept of a *physical theory* is then that of a theory which describes the conjectured relations between observables in terms of axioms relating these propositions. The concept of space determined by the observables concerned ought then to be that of the algebra of propositions in this theory. The question, then, is, in what logical context is this Lindenbaum algebra of propositions to be calculated: in other words, with what rules of deduction are physical observations naturally manipulated. One observation is immediately evident, namely, that the logic is necessarily more like intuitionistic logic than classical logic, corresponding to a rejection of the excluded middle, at least in its generality, by physical scientists. Moreover, it can be seen that a fundamental role is enjoyed by a logical operation

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which retains a vestige of temporality in its interpretation: namely, that this operation is not assumed to be commutative, corresponding to the quantum mechanical notion that the order of making observations is central to their description.

3.

In extending Gelfand duality to the constructive context, Banaschewski and I observed in 1980 [2] the importance of exactly these propositions,

$$a \in (r,s)$$

for each $a \in A$ and each rational open rectangle (r,s) in the complex plane, in axiomatising the locale which is the maximal spectrum of a commutative C*-algebra A , and in obtaining the Gelfand representation of A . This observation suggests an intimate connection between the questions posed above, concerning the Gelfand-Naimark representation of a

non-commutative C*-algebra and concerning the foundations of quantum mechanics in terms of propositional theories within a constructive, but non-commutative, logic, perhaps more fundamental than the formalistic links to C*-algebras within the conventional foundation of quantum mechanics by way of Hilbert space. This leads one to examine the possibility that the Gelfand-Naimark representation in the non-commutative case might be obtained almost immediately by reconsidering the axiomatisation of the maximal spectrum within a non-commutative logic.

An identification of the rules of deduction appropriate to the physical situation might therefore be expected to be equivalent to that of those within which the theory of the maximal spectrum must be considered in the non-commutative case.

This interdependence is made yet more intriguing by the observation that in the set theories based on quantum logics considered by Takeuti [7] the complex numbers are non-commutative, allowing the possibility that a Gelfand-Naimark representation may exist which is extremely closely linked to that of a commutative C*-algebra.

It is now conjectured that the logic which arises in these situations is that characterised by the following definition:

> by a *quantale* will be meant a complete lattice together with an associative product $\bullet$ which is distributive on both sides over arbitrary joins $\vee$.

The lattice of closed left ideals of a C*-algebra is a quantale with respect to multiplication of left ideals. In this situation, the identity element of the lattice is a left identity, but not generally a right identity, with respect to the product. Although the maximal spectrum of a C*-algebra appeared above to consist in the covering of the quantale $\mathrm{Lid}\,A$ by the quantale $\mathrm{Lid}\,B$ determined by its irreducible representations, it may be conjectured that the lattice $\mathrm{Lid}\,B$ is no longer necessary to its description once the quantale structure of $\mathrm{Lid}\,A$ is taken into account. The role of $\mathrm{Lid}\,B$ appears to have been that of defining the product on the lattice $\mathrm{Lid}\,A$ in terms of that definable canonically on $\mathrm{Lid}\,B$ in terms of its lattice structure. Moreover, it may be conjectured that the quantale $\mathrm{Lid}\,A$ may be obtained from a theory involving propositions

$$a \in (r,s)$$

with axioms not differing substantially from those in the commutative case, within rules of deduction derived from the axioms of a quantale in the same way that those of intuitionistic propositional logic derive from those of a locale. Logically, the product $\bullet$ now interprets the connective $\&$ denoting sequential conjunction, no longer assumed to be commutative (cf. [3]), while the join interprets conventionally the arbitrary disjunction $\vee$, allowing operations of left and right implication to exist distinctly.

It may be remarked that the straightforward development of these logical aspects does appear to need the identity element of the lattice to be a left identity with respect to the product of the quantale, which is indeed the case for the quantale of closed left ideals of a C*-algebra. The apparent asymmetry of this requirement just reflects that inherent in having to write mappings on a particular side of their argument. Indeed, to attain the natural interpretation of this logical notation, particularly in respect of that needed in considering foundations of quantum mechanics within this framework, it becomes necessary to write mappings to the right, investigating instead lattices of closed *right* ideals of C*-algebras and quantales in which the identity element is a *right* identity with respect to the product.

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References

1. Akemann, C.A.: Left ideal structure of C*-algebras, J. Functional Analysis, **6** (1970), 305-317.

2. Banaschewski, B., Mulvey, C.J.: Gelfand duality in a Grothendieck topos. *To appear*.

3. Dilworth, R., Ward, M.: Residuated lattices. Trans. Amer. Math. Soc., **45** (1939), 335-354.

4. Giles, R., Kummer, H.: A non-commutative generalisation of topology. Indiana Math. J., **21** (1971), 91-102.

5. Mulvey, C.J.: Are there sheaves in the foundations of physics? Lecture, University of Oxford, 1975.

6. Mulvey, C.J.: &. Tagungsbericht, Oberwolfach Category Meeting, 1983.

7. Takeuti, G.: Quantum set theory. Preprint, University of Illinois, 1981.