Construction of a local and global Lyapunov Function for Discrete Dynamical Systems using Radial Basis Functions

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June 4, 2007

Abstract

The basin of attraction of an asymptotically stable fixed point of the discrete dynamical system given by the iteration $x_{n+1} = g(x_n)$ can be determined through sublevel sets of a Lyapunov function. In [3] a Lyapunov function is constructed by approximating the solution of a difference equation using radial basis functions. However, the resulting Lyapunov function is non-local, i.e. it has no negative discrete orbital derivative in a neighborhood of the fixed point. In this paper we modify the construction method by using the Taylor polynomial and thus obtain a Lyapunov function with negative discrete orbital derivative both locally and globally.


1 Introduction

In this paper we study the basin of attraction of a fixed point of the general discrete dynamical system given by the iteration $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geq 1$ and $d \in \mathbb{N}$. While existence and exponential asymptotic stability of a fixed point can be checked straight-forward, the determination of its basin of attraction $A(\overline{x})$ consisting of all initial points such that their iterates converge to $\overline{x}$, is a difficult problem, since it involves global information as opposed to local information for the stability. A common way to analyze the basin of attraction is the method of a Lyapunov function.

A Lyapunov function $L \in C^0(\mathbb{R}^d, \mathbb{R})$ is a function with negative discrete orbital derivative $L'(x) < 0$ for all $x \in B \setminus \{\overline{x}\}$, where $B$ is an open neighborhood of the fixed point. The discrete orbital derivative is defined by $L'(x) := L(g(x)) - L(x)$. Then connected and bounded sublevel sets $O_R$, i.e. $L(x) < R$ for all $x \in O_R$ and $L(x) = R$ for all $x \in \partial O_R$, are subsets of the basin of attraction of $\overline{x}$ supposed that $L'(x) < 0$ holds for all $x \in \overline{O_R} \setminus \{\overline{x}\}$.

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The existence of Lyapunov functions has been established by converse theorems which, however, offer no construction method for Lyapunov functions. One of these converse theorems proves the existence of the Lyapunov function $V \in C^\sigma(A(\mathfrak{F}), \mathbb{R})$ satisfying $V'(x) = -\|x - \mathfrak{F}\|^2$, cf. [7] or [3], Theorem 2.8. In [3], this function $V$ was approximated as the solution of a linear difference equation using radial basis functions. For the approximation $v$ one chooses the coefficients of a certain ansatz function such that the difference equation is satisfied for all points of a grid $X_N$, i.e. $v'(x_j) = -\|x_j - \mathfrak{F}\|^2$ for all $x_j \in X_N$. The approximation then satisfies the error bound $|V'(x) - v'(x)| \leq \epsilon$, i.e. $v'(x) \leq -\|x - \mathfrak{F}\|^2 + \epsilon$, where $\epsilon$ depends on the density of the grid. This ensures $v'(x) < 0$ if $x \in B_{\sqrt{\mathfrak{F}}(\mathfrak{F})}$, i.e. we obtain a non-local Lyapunov function. The local part was solved using a local Lyapunov function, which can be obtained by a Lyapunov function of the linearized equation $x_{n+1} = Dg(\mathfrak{F})x_n$.

In this paper, we seek to construct a Lyapunov function $v$ which has negative discrete orbital derivative also near $\mathfrak{F}$ and thus is local and global. To achieve this goal, we calculate a Taylor polynomial-like function $n(x)$ of $V$, such that the function $W(x) := \frac{V(x)}{n(x)}$ for $x \in A(\mathfrak{F}) \setminus \{\mathfrak{F}\}$ and $W(\mathfrak{F}) := 1$ is a smooth function. Then we approximate the function $W(x)$ satisfying the linear difference equation $n'(x)W(x) + n(x)W'(x) = -\|x - \mathfrak{F}\|^2$ by an approximation $w$, which satisfies this equation for all points of a grid and, additionally, $w(\mathfrak{F}) = 1$. The radial basis function chosen for this mixed approximation will be of the class of Wendland’s compactly supported radial basis functions. Local and global error estimates of the approximation ensure that $v(x) = n(x)w(x)$ has negative discrete orbital derivative in particular near $\mathfrak{F}$, i.e. $v$ is a local and global Lyapunov function. Hence, connected sublevel sets of $v$ are subsets of the basin of attraction of $\mathfrak{F}$. We prove in this paper that each open, bounded and connected subset of the basin of attraction can be covered by a sublevel set of $v$, when choosing the grid points properly.

For continuous dynamical systems given by an autonomous ordinary differential equation $\dot{x} = f(x)$ a similar method of constructing Lyapunov functions was established in [2]; improved error estimates were obtained in [5]. The construction of a global and local Lyapunov function similar to this paper was presented in [4].

The paper is organized as follows: in Section 2 we recall the properties of the Lyapunov function $V$ and prove properties of its Taylor polynomial-like function $n(x)$ and the function $W(x) = \frac{V(x)}{n(x)}$. Section 3 deals with the approximation $w$ of $W$ using radial basis functions. In Section 4 we derive local and global error estimates for $w$, which are used in Section 5 to prove the main results: setting $v(x) = n(x)w(x)$, the function $v$ is a local and global Lyapunov function, i.e. $v'(x) < 0$ holds for all $x \in B \setminus \{\mathfrak{F}\}$, cf. Theorem 5.1, and each open, bounded and connected subset of the basin of attraction can be covered by a sublevel set of $v$, cf. Theorem 5.2. In Section 6 we give two examples: a comparison with the method in [3] (two-dimensional example) and a model for the demand for education, cf. [9] (three-dimensional example). In the appendix we provide the formula of the function $n$ for Section 6.2 as well as the proof of Lemma 2.4.
2 The Lyapunov Function $V$ and its Taylor polynomial

Throughout the paper we consider the discrete dynamical system given by the iteration

$$x_{n+1} = g(x_n)$$

where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geq 1$ and $n \in \mathbb{N}$. The flow of the dynamical system is denoted by $S_n x_0 := x_n$, $n \in \mathbb{N}_0$.

We assume that $\overline{\mathcal{X}} \in \mathbb{R}^d$ is a fixed point and that all eigenvalues $\lambda$ of $Dg(\overline{\mathcal{X}})$ satisfy $|\lambda| < 1$, i.e. $\overline{\mathcal{X}}$ is exponentially asymptotically stable. In the following theorem a sufficient condition for the determination of a subset of the basin of attraction is given through the Lyapunov function $L$, cf. [6], [7] or [3], Theorem 2.2.

**Theorem 2.1** Let $\overline{\mathcal{X}}$ be an asymptotically stable fixed point of the discrete dynamical system $x_{n+1} = g(x_n)$, let $\mathcal{X} \in O \subset \mathbb{R}^d$ be an open, bounded and connected set. Let $L \in C^0(\mathbb{R}^d, \mathbb{R})$ be a function and $R^* \in \mathbb{R}$ such that

1. $L(x) < R^*$ holds for all $x \in O$ and $L(x) = R^*$ holds for all $x \in \partial O$.
2. $L'(x) < 0$ holds for all $x \in \mathcal{O} \setminus \{\mathcal{X}\}$, where $L'(x) := L(g(x)) - L(x)$ denotes the discrete orbital derivative.

Then $\mathcal{O}$ is positively invariant and $\mathcal{O} \subset A(\overline{\mathcal{X}})$ holds.

The following theorem ensures the existence and smoothness of a special Lyapunov function, cf. Theorem 49.3 of [7] for the existence and Theorem 2.8 of [3] with $p(x) = -\|x - \overline{\mathcal{X}}\|^2$ for the smoothness.

**Theorem 2.2 (Existence of $V$)** Let $\overline{\mathcal{X}}$ be a fixed point of $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geq 0$ and $g$ is $C^1$ in a neighborhood of $\overline{\mathcal{X}}$. Let $|\lambda| < 1$ hold for all eigenvalues of $Dg(\overline{\mathcal{X}})$.

Then there exists a function $V \in C^\sigma(A(\overline{\mathcal{X}}), \mathbb{R})$ with $V(\overline{\mathcal{X}}) = 0$ such that

$$V'(x) = -\|x - \overline{\mathcal{X}}\|^2$$

holds for all $x \in A(\overline{\mathcal{X}})$. The set $K_R := \{x \in A(\overline{\mathcal{X}}) \mid V(x) \leq R\}$ is compact in $\mathbb{R}^d$ for all $R \geq 0$.

In the following Definition 2.3 we define the function $\eta$ which turns out to be the Taylor polynomial of $V$ of order $P$, cf. Lemma 2.4.

**Definition 2.3 (Definition of $\eta$)** Let $\sigma \geq P \geq 2$. Let $\eta$ be the function

$$\eta(x) := \sum_{2 \leq |\alpha| \leq P} \frac{c_\alpha}{\alpha!} (x - \overline{\mathcal{X}})^\alpha$$

(2.1)

such that $\eta'(x) = \eta(g(x)) - \eta(x) = -\|x - \overline{\mathcal{X}}\|^2 + o(\|x - \overline{\mathcal{X}}\|^P)$. (2.2)
Lemma 2.4 There is one and only one function \( h \) of the form (2.1) which satisfies (2.2). The function \( h \) is the Taylor polynomial of \( V \), cf. Theorem 2.2, of order \( P \).

The proof of the lemma can be found in Appendix B. Let us explain how to calculate the function \( h \).

Remark 2.5 The equation (2.2) can be solved by plugging the ansatz (2.1) into (2.2) and replacing \( g \) by its Taylor polynomial of order \( P - 1 \). Then (2.2) becomes

\[
\sum_{2 \leq |\alpha| \leq P} c_{\alpha} \left[ \left( Dg(\overline{\pi})(x - \overline{\pi}) + \sum_{2 \leq |\beta| \leq P-1} \frac{\partial^\beta g(\overline{\pi})}{\beta!} (x - \overline{\pi})^\beta \right)^\alpha - (x - \overline{\pi})^\alpha \right] = -\|x - \overline{\pi}\|^2 + o(\|x - \overline{\pi}\|^P). \tag{2.3}
\]

The difference of \( g \) to its Taylor polynomial is of order \( o(\|x - \overline{\pi}\|^P) \) and thus the difference of the whole left-hand side of (2.3) to \( h'(x) \) is of order \( o(\|x - \overline{\pi}\|^P) \).

We add a high order polynomial \( M\|x - \overline{\pi}\|^{2H} \) to \( h \) in order to obtain a function \( n \) for which \( n(x) > 0 \) holds for all \( x \neq \overline{\pi} \).

Definition 2.6 (Definition of \( n \)) Let \( \sigma \geq P \geq 2 \) and let \( h \) be as in Definition 2.3. Let \( H := \left\lceil \frac{P}{2} \right\rceil + 1 \) and \( M \geq 0 \) and define

\[
n(x) = h(x) + M\|x - \overline{\pi}\|^{2H} = \sum_{2 \leq |\alpha| \leq P} c_{\alpha} (x - \overline{\pi})^\alpha + M\|x - \overline{\pi}\|^{2H}.
\tag{2.4}
\]

Choose the constant \( M \) so large, that \( n(x) > 0 \) holds for all \( x \neq \overline{\pi} \). Note that \( n \in C^\infty(\mathbb{R}^d, \mathbb{R}) \).

This is possible since for \( x \to \overline{\pi} \) the leading term is \( h_2(x) = (x - \overline{\pi})^T B(x - \overline{\pi}) \) which is a positive definite quadratic form, and for \( x \to \infty \) the leading term is \( M\|x - \overline{\pi}\|^{2H} \); for details cf. [2], Definition 2.56.

In Proposition 2.7 we summarize some properties of \( n(x) \) and \( W(x) := \frac{V(x)}{n(x)} \).

Proposition 2.7 Let \( \sigma \geq P \geq 2 \) and let \( n \) be as in Definition 2.6. Then

1. \( n(x) > 0 \) holds for all \( x \neq \overline{\pi} \).

2. For each compact set \( K \), there is a constant \( C > 0 \) such that \( n(x) \leq C\|x - \overline{\pi}\|^2 \) holds for all \( x \in K \).

3. \( W(x) = \frac{V(x)}{n(x)} \in C^{P-2}(A(\overline{\pi}), \mathbb{R}) \), where \( W(\overline{\pi}) = 1 \), and \( \partial^\alpha_x W(\overline{\pi}) = 0 \) for all \( 1 \leq |\alpha| \leq P - 2 \).

Proof: The proposition can be proved as in the continuous case, cf. [2], Proposition 2.58. \( \square \)
Example 2.8 Consider the difference equation

\[
\begin{align*}
    x_{n+1} &= \frac{1}{2}x_n + x_n^2 - y_n^2 \\
y_{n+1} &= -\frac{1}{2}y_n + x_n^2
\end{align*}
\]  

(2.5)

with fixed point \( x = (0,0) \). This is Example 1 of [3] and will serve as an example in Section 6.1. We calculate the polynomial \( h(x,y) \) and start with the terms \( h_2(x,y) \) of second order. By the proof of Lemma 2.4, \( h_2(x,y) = (x, y)B \left( \begin{array}{c} x \\ y \end{array} \right) \), where \( B \)

is the solution of \( Dg(0,0)^TBDg(0,0) - B = -I \). In our case \( B = \left( \begin{array}{cc} 0 & \frac{4}{3} \\ 0 & \frac{4}{3} \end{array} \right) \), i.e. \( h_2(x,y) = \frac{4}{3}x^2 + \frac{4}{3}y^2 \), cf. [3].

For the terms of order three, \( h_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \), we calculate the left-hand side of (2.2) with \( P = 3 \)

\[-x^2 - y^2 + o(||(x,y)||^3)\]

\[= (h_2 \circ g)(x, y) + (h_3 \circ g)(x, y) - h_2(x, y) - h_3(x, y)\]

\[= \frac{4}{3}\left(\frac{1}{2}x + x^2 - y^2\right)^2 + \frac{4}{3}\left(-\frac{1}{2}y + x^2\right)^2 + a\left(\frac{1}{2}x + x^2 - y^2\right)^3\]

\[+b\left(\frac{1}{2}x + x^2 - y^2\right)^2\left(-\frac{1}{2}y + x^2\right) + c\left(\frac{1}{2}x + x^2 - y^2\right)\left(-\frac{1}{2}y + x^2\right)^2\]

\[+d\left(-\frac{1}{2}y + x^2\right)^3 - \frac{4}{3}x^2 - \frac{4}{3}y^2 - ax^3 - bx^2y - cxy^2 - dy^3\]

\[= -x^2 - y^2 + x^3\left(\frac{4}{3} + \frac{1}{8}a - a\right) + x^2y\left(-\frac{4}{3} + \frac{1}{8}b - b\right) + xy^2\left(-\frac{4}{3} + \frac{1}{8}c - c\right)\]

\[+y^3\left(-\frac{1}{8}d - d\right) + o(||(x,y)||^3),\] i.e.

\[a = \frac{32}{27}, \quad b = -\frac{32}{27}, \quad c = -\frac{32}{27} \quad \text{and} \quad d = 0.\] Thus, \( h_3(x,y) = \frac{32}{27}x^3 - \frac{32}{27}x^2y - \frac{32}{27}xy^2.\)

In a similar way we calculate the terms of order 4 and 5 and obtain for \( P = 5 \)

\[h(x,y) = \frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{32}{21}x^3 - \frac{32}{21}x^2y - \frac{32}{21}xy^2\]

\[+ \frac{10624}{2835}x^4 + \frac{4096}{3213}x^3y - \frac{1408}{315}x^2y^2 - \frac{256}{459}xy^3 - \frac{64}{35}y^4\]

\[+ \frac{3657728}{530145}x^5 + \frac{1416704}{530145}x^4y - \frac{65536}{9639}x^3y^2 - \frac{1581056}{530145}x^2y^3 + \frac{34304}{9765}xy^4\]

\[+ \frac{2560}{5049}y^5.\]

(2.6)

A function \( n(x,y) > 0 \) holds for all \( (x,y) \neq (0,0) \) is given by

\[n(x,y) = h(x,y) + 2(x^2 + y^2)^3.\]

(2.7)
3 Approximation of $W$ using Radial Basis Functions

In this section we describe the approximation of the function $W$ by $w$ using radial basis functions. For an overview on radial basis function, cf. [12] or [1]. Recall that $V(x) = n(x)W(x)$ and $V'(x) = -\|x - \overline{x}\|^2$. Hence, the function $W$ satisfies the equation $n(g(x))W(g(x)) - n(x)W(x) = -\|x - \overline{x}\|^2$, or in other words

$$D_nW(x) = -\|x - \overline{x}\|^2$$

(3.1)

for all $x \in A(\overline{x})$, where $D_nW(x) := n(g(x))W(g(x)) - n(x)W(x)$ is a linear operator.

For the approximating function $w : \mathbb{R}^d \to \mathbb{R}$ we choose a radial basis function $\Psi(x) := \psi(\|x\|)$ and a grid $X_N \subset \mathbb{R}^d$, and we make the following ansatz:

$$w(x) = \beta_0\Psi(x - \overline{x}) + \sum_{k=1}^{N} \beta_k(\delta_{x_k} \circ D_n)^y\Psi(x - y),$$

(3.2)

where $\beta_k \in \mathbb{R}$ and $\delta$ denotes Dirac’s $\delta$-distribution, i.e. $\delta_x f(x) = f(x^*)$. The advantage of this ansatz including the operator $D_n$ is, as it turns out, that the interpolation matrix $A$ will be symmetric. The ansatz is a mixed approximation including the linear operators $D_n$ for the grid points $x_1, \ldots, x_N$ and the linear operator $D^0 = \text{id}$ for the grid point $\overline{x}$.

The coefficients $\beta_k$ are determined by the claim that

$$w(\overline{x}) = W(\overline{x}) = 1$$

(3.3)

and $(\delta_{x_j} \circ D_n)^y w(x) = (\delta_{x_j} \circ D_n)^y W(x)$

(3.4)

holds for all $j = 1, \ldots, N$, i.e. $D_nW(x_j) = D_nw(x_j)$. Plugging the ansatz (3.2) into (3.3) and (3.4) one obtains with (3.1)

$$1 = \beta_0\Psi(0) + \sum_{k=1}^{N} \beta_k(\delta_{x_k} \circ D_n)^y\Psi(\overline{x} - y)$$

and

$$-\|x_j - \overline{x}\|^2 = (\delta_{x_j} \circ D_n)^y W(x_j)
= (\delta_{x_j} \circ D_n)^y w(x)
= (\delta_{x_j} \circ D_n)^y \Psi(x - \overline{x})\beta_0 + \sum_{k=1}^{N} (\delta_{x_j} \circ D_n)^y(\delta_{x_k} \circ D_n)^y\Psi(x - y)\beta_k$$

for all $j = 1, \ldots, N$, since $D_n$ is a linear operator. This is equivalent to the following system of linear equations with interpolation matrix $A = (a_{jk})_{j,k=0,\ldots,N}$ and vector $\alpha = (\alpha_j)_{j=0,\ldots,N}$, cf. also Proposition 3.5.

$$A\beta = \alpha,$$

where

$$a_{jk} = (\delta_{x_j} \circ D_n)^y(\delta_{x_k} \circ D_n)^y\Psi(x - y) \text{ for } j, k \geq 1$$

(3.5)

$$a_{0k} = a_{k0} = (\delta_{x_k} \circ D_n)^y\Psi(\overline{x} - y) \text{ for } k \geq 1$$

(3.6)

$$a_{00} = \Psi(0)$$

(3.7)

and

$$\alpha_j = -\|x_j - \overline{x}\|^2 \text{ for } j \geq 1$$

$$\alpha_0 = 1.$$
A clearly is a symmetric matrix. For existence and uniqueness of the solution \(\beta\), the interpolation matrix \(A\) must have full rank. We will even obtain a positive definite matrix \(A\) for grids which include no fixed or periodic point, cf. Proposition 3.4.

The radial basis functions \(\Psi\) used in this paper will be from the class of Wendland functions which were introduced by Wendland, cf. [11]. They have compact support and are polynomials on their support.

**Definition 3.1 (Wendland functions, [11])** Let \(l \in \mathbb{N}, k \in \mathbb{N}_0\). We define by recursion \(\psi_{l,0}(r) = (1-r)^l_+\) and \(\psi_{l,k+1}(r) = \int_0^1 t \psi_{l,k}(t) \, dt\) for \(r \in \mathbb{R}_0^+\). Here we set \(x_+ = x\) for \(x \geq 0\) and \(x_+ = 0\) for \(x < 0\).

For the rest of the paper we define \(\Psi\) by a Wendland function \(\psi_{l,k}\) with \(k \geq 1\).

**Assumptions** We define the radial basis function \(\Psi\) by
\[
\Psi(x) := \psi_{l,k}(c\|x\|),
\]
where \(\psi_{l,k}\) is a Wendland function, \(k \in \mathbb{N}, l := \left\lfloor \frac{d}{2} \right\rfloor + k + 1\) and \(c > 0\). Denote \(\psi(r) := \psi_{l,k}(cr)\).

In Proposition 3.2 we summarize important properties of \(\psi_{l,k}(r)\) and \(\Psi(x)\). For a proof, cf. [11] and [2].

**Proposition 3.2** Let \(k \in \mathbb{N}\) and \(l := \left\lfloor \frac{d}{2} \right\rfloor + k + 1\). Let \(\psi(r) := \psi_{l,k}(cr)\) with \(c > 0\) and \(\Psi(x) := \psi(\|x\|)\). Moreover, define \(\psi_1(r) := \frac{d}{dr} \psi(r)\) for \(r > 0\) and \(\psi_2(r) := \frac{d}{dr} \psi_1(r)\) for \(r > 0\) and set \(\psi_2(0) = 0\). Set
\[
\mathcal{F}^* = H^{-\left(\frac{d+1}{2}+k\right)}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{F} = H^{\frac{d+1}{2}+k}(\mathbb{R}^d),
\]
where \(H\) denotes the Sobolev space. Then

1. \(\Psi \in C^{2k}(\mathbb{R}^d, \mathbb{R})\) and \(\Psi\) has compact support.

2. For \(\psi(r) := \psi_{l,k}(cr)\) we have the following asymptotics for the functions \(\psi_1\) and \(\psi_2\), respectively:
   - \(\frac{d}{dr} \psi(r) = O(r)\) for \(r \to 0\).
   - \(\psi_1(r) = O(1)\) for \(r \to 0\) and \(\lim_{r \to 0} \psi_1(r) =: \psi_1(0)\) exists.
   - \(\frac{d}{dr} \psi_1(r) = O(1)\) for \(r \to 0\).
   - \(\psi_2(r) = O(\frac{1}{r})\) for \(r \to 0\).

3. For the Fourier transform \(\tilde{\Psi}(\omega) = \int_{\mathbb{R}^d} \Psi(x)e^{-ix^T \omega} \, dx\) we have
\[
C_1 (1 + \|\omega\|^2)^{-\left(\frac{d+1}{2}+k\right)} \leq \tilde{\Psi}(\omega) \leq C_2 (1 + \|\omega\|^2)^{-\left(\frac{d+1}{2}+k\right)}
\]
with positive constants \(C_1, C_2\).
4. Let $\lambda \in F^* \subset S'(\mathbb{R}^d)$, where $S'(\mathbb{R}^d)$ denotes the dual of the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions. Then we define the norm $||\lambda||_{F^*}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\lambda}(\omega)^2 \Psi(\omega) \ d\omega$, which is equivalent to the usual Sobolev norm in $F^* = H^{-\left(\frac{d+1}{2}+k\right)}(\mathbb{R}^d)$. Moreover, $||\lambda||_{F^*}^2 = \lambda^T \overline{\lambda} \Psi(x-y)$.

5. If $\lambda \in F^*$, then $\lambda * \Psi \in F$.

In the following lemma we show that several operators are in $F^*$ and, moreover, the approximating function $w$ is an element of the function space $F$.

**Lemma 3.3** Let the linear operator $D_n$ be given by $D_n w(x) := n(g(x))w(g(x)) - n(x)w(x)$ and let the linear operator $D$ of the discrete orbital derivative be given by $Dw(x) := w(g(x)) - w(x)$. Denote by $E'(\mathbb{R}^d)$ the distributions with compact support.

For all $x \in \mathbb{R}^d$ we have $\delta_x \circ D, \delta_x \circ D_n \in F^* \cap E'(\mathbb{R}^d)$. Moreover, for each grid $X_N$ and any $\beta_0, \beta_1, \ldots, \beta_N \in \mathbb{R}$ the ansatz function $w$ of (3.2) satisfies $w(x) = \beta_0 \Psi(x - \tau) + \sum_{k=1}^N \beta_k (\delta_{x_k} \circ D_n)^{\Psi}(x-y) \in F$.

**Proof:** To prove that a linear operator belongs to $F^*$ we show that $\int_{\mathbb{R}^d} \hat{\lambda}(\omega)^2 \Psi(\omega) \ d\omega < \infty$, cf. 4. of Proposition 3.2. Note that $\Psi(\omega) \leq C_2 \left( 1 + ||\omega||^2 \right)^{-\left(\frac{d+1}{2}+k\right)}$ by (3.10).

Since $\delta_x$ has compact support, $\hat{\delta}_x(\omega) = \delta_x e^{ix^T \omega}$. Thus,

$$\int_{\mathbb{R}^d} |\hat{\delta}_x(\omega)|^2 \Psi(\omega) \ d\omega \leq C_2 \int_{\mathbb{R}^d} (1 + ||\omega||^2)^{-\left(\frac{d+1}{2}+k\right)} \ d\omega < \infty.$$

Since $\delta_x \circ D$ has compact support, $(\delta_x \circ D)^{\hat{\Psi}}(\omega) = (\delta_x \circ D)e^{ix^T \omega} = e^{ig(x)^T \omega} - e^{ix^T \omega}$. Thus,

$$\int_{\mathbb{R}^d} |(\delta_x \circ D)^{\hat{\Psi}}(\omega)|^2 \Psi(\omega) \ d\omega \leq 4C_2 \int_{\mathbb{R}^d} (1 + ||\omega||^2)^{-\left(\frac{d+1}{2}+k\right)} \ d\omega < \infty.$$

Since $\delta_x \circ D_n$ has compact support, $(\delta_x \circ D_n)^{\hat{\Psi}}(\omega) = (\delta_x \circ D_n)e^{ix^T \omega} = n(g(x))e^{ig(x)^T \omega} - n(x)e^{ix^T \omega}$. Thus,

$$\int_{\mathbb{R}^d} |(\delta_x \circ D_n)^{\hat{\Psi}}(\omega)|^2 \Psi(\omega) \ d\omega \leq 4C_2 \max(n(g(x)), n(x))^2 \int_{\mathbb{R}^d} (1 + ||\omega||^2)^{-\left(\frac{d+1}{2}+k\right)} \ d\omega < \infty.$$

Let $\lambda = \beta_0 \delta_\tau + \sum_{j=1}^N \beta_j (\delta_{x_j} \circ D_n)$. Since $\lambda \in S'(\mathbb{R}^n)$ has the compact support $\text{supp}(\lambda) = \bigcup_{j=1}^N \{x_j\} \cup \bigcup_{j=1}^N \{g(x_j)\} \cup \{\tau\} =: K$, we have $\lambda * \Psi(x) = \lambda^T \Psi(x-y) = w(x)$. We conclude $\lambda \in F^* \cap E'(\mathbb{R}^d)$ in a similar way as above. This implies $w = \lambda * \Psi \in F$ by 5. of Proposition 3.2. \hfill \Box

Now we show that the collocation matrix $A$ is positive definite.
**Proposition 3.4** Let $\Psi(x)$ be as in the Assumptions and let $X_N = \{x_1, x_2, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Then the matrix $A$ defined in (3.5) to (3.7) is positive definite.

**Proof:** For $\lambda = \beta_0 \delta_x + \sum_{j=1}^N \beta_j (\delta_{x_j} \circ D_n) \in \mathcal{F} \cap \mathcal{E}'(\mathbb{R}^d)$, cf. Lemma 3.3, we have with 4. of Proposition 3.2 $\beta^T A \beta = \lambda^T \overline{\lambda} \overline{\Psi}(x - y) = \|\lambda\|_{\mathcal{F}}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\lambda}(\omega)|^2 \hat{\Psi}(\omega) d\omega$. Since $\hat{\Psi}(\omega) > 0$ holds for all $\omega \in \mathbb{R}^d$, the matrix $A$ is positive semidefinite.

Now we show that $\beta^T A \beta = 0$ implies $\beta = 0$. We assume that $\beta^T A \beta = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\lambda}(\omega)|^2 \hat{\Psi}(\omega) d\omega = 0$. Then the analytic function satisfies $\hat{\lambda}(\omega) = 0$ for all $\omega \in \mathbb{R}^d$. By Fourier transformation in $\mathcal{S}'(\mathbb{R}^d)$ we have $\mathcal{S}'(\mathbb{R}^d) \ni \lambda = 0$, i.e.

$$\lambda(h) = \beta_0 h(\overline{x}) + \sum_{j=1}^N \beta_j [n(g(x_j)) h(g(x_j)) - n(x_j) h(x_j)] = 0 \quad (3.11)$$

for all test functions $h \in \mathcal{S}(\mathbb{R}^d)$.

We test that $\beta_0 = 0$. The points $x_j, j = 1, \ldots, N$ are distinct from the fixed point $\overline{x}$. Denote $I := \{j \in \{1, \ldots, N\} \mid g(x_j) \neq \overline{x}\}$. There is a neighborhood $B_\delta(\overline{x})$ such that $x_j \notin B_\delta(\overline{x})$ holds for all $j = 1, \ldots, N$ and $g(x_i) \notin B_\delta(\overline{x})$ holds for all $i \in I$. Define the function $h(x) = 1$ for $x \in B_{\frac{\delta}{2}}(\overline{x})$ and $h(\overline{x}) = 0$ for $x \notin B_{\frac{\delta}{2}}(\overline{x})$, and extend it smoothly such that $h \in \mathcal{S}(\mathbb{R}^d)$. Then (3.11) yields $0 = \lambda(h) = \beta_0$ since for $i \notin I$ we have $n(g(x_i)) h(g(x_i)) = n(\overline{x}) h(\overline{x}) = 0$.

Now we show $\beta_i = 0$ for $i = 1, \ldots, N$. Fix an $i \in \{1, \ldots, N\}$. Denote $J := \{j \in \{1, \ldots, N\} \setminus \{i\} \mid \exists k \in \mathbb{N} : g^k x_j = x_i \text{ and } \forall l = 1, \ldots, k - 1 : g^l x_j \notin X_N\}$. Note that $g(x_k) \in \{x_j \mid j \in J \cup \{i\}\}$ for all $k \in J$ by definition of $J$ and $g(x_i) \notin \{x_j \mid j \in J \cup \{i\}\}$, since $x_i$ is no fixed point and no periodic point. Moreover, by definition of $J$ we have

$$g(x_k) \notin \{x_j \mid j \in J \cup \{i\}\} \text{ for all } k \in \{1, \ldots, N\} \setminus (J \cup \{i\}). \quad (3.12)$$

Define the function $h \in \mathcal{S}(\mathbb{R}^d)$ in the following way:

$$h(x_j) = \begin{cases} 1/n(x_j) & \text{for } j \in J \cup \{i\} \\ 0 & \text{for } j \in \{1, \ldots, N\} \setminus (J \cup \{i\}) \end{cases}, \quad h(g(x_j)) = 0 \text{ for } j \in \{1, \ldots, N\} \setminus J,$$

$$h(\overline{x}) = 0.$$

Note that $n(x) \neq 0$ for $x \neq \overline{x}$. Extend the function smoothly and prolongate it by zero such that $h \in \mathcal{S}(\mathbb{R}^d)$. Note that this is possible by (3.12).

Then (3.11) yields

$$0 = \lambda(h) = 0 + \sum_{j \in J} \beta_j [1 - 1] + \beta_i [0 - 1] + \sum_{k \in \{1, \ldots, N\} \setminus (J \cup \{i\})} \beta_k [0 - 0] = -\beta_i.$$

This argumentation holds for all $i = 1, \ldots, N$ and thus $\beta = 0$. Hence, $A$ is positive definite. \qed
In the following proposition we calculate the matrix elements of the collocation matrix $A$, cf. (3.5) to (3.7).

**Proposition 3.5** Let $X_N = \{x_1, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Let $\Psi(x) := \psi(||x||)$ be defined by a Wendland function as in the Assumptions.

The matrix elements $a_{jk}$ of the collocation matrix $A$ are then given by

$$a_{jk} = n(g(x_j))n(g(x_k))\psi(||g(x_j) - g(x_k)||) - n(g(x_j))n(x_k)\psi(||g(x_j) - x_k||)$$

\[ - n(x_j)n(g(x_k))\psi(||x_j - g(x_k)||) + n(x_j)n(x_k)\psi(||x_j - x_k||), \quad (3.13) \]

$$a_{k0} = a_{0k} = n(g(x_k))\psi(||x - g(x_k)||) - n(x_k)\psi(||x - x_k||), \quad (3.14)$$

$$a_{00} = \psi(0) \quad (3.15)$$

for $j, k = 1, \ldots, N$.

The vector $\alpha$ is given by

$$\alpha_j = -||x_j - \overline{x}||^2 \text{ for } j \geq 1,$$

$$\alpha_0 = 1.$$

Let the vector $\beta = (\beta_0, \beta_1, \ldots, \beta_N)^T$ be the unique solution of the system of linear equations $A\beta = \alpha$.

Then the approximant $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ is given by

$$w(x) = \beta_0\psi(||x - \overline{x}||)$$

$$+ \sum_{k=1}^{N} \beta_k [n(g(x_k))\psi(||x - g(x_k)||) - n(x_k)\psi(||x - x_k||)].$$

**Proof:** By (3.5) we have for $j, k \geq 1$

\[ a_{jk} = (\delta_{x_j} \circ D_n)^x (\delta_{x_k} \circ D_n)^y \psi(||x - y||) \]

\[ = (\delta_{x_j} \circ D_n)^x [n(g(x_k))\psi(||x - g(x_k)||) - n(x_k)\psi(||x - x_k||)] \]

\[ = n(g(x_j))n(g(x_k))\psi(||g(x_j) - g(x_k)||) - n(g(x_j))n(x_k)\psi(||g(x_j) - x_k||) \]

\[ - n(x_j)n(g(x_k))\psi(||x_j - g(x_k)||) + n(x_j)n(x_k)\psi(||x_j - x_k||). \]

The formulas for $j = 0, k = 0$ and $w(x)$ follow by similar calculations, cf. also (3.2).

The following proposition is the minimality property of the approximation $w$; it can be proved in a similar way as e.g. [2], Proposition 3.34.

**Proposition 3.6** Let $\Psi(x)$ be defined as in the Assumptions. Let $X_N = \{x_1, x_2, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Let $W \in \mathcal{F}$ and let $w \in \mathcal{F}$, cf. Lemma 3.3, be the approximation of $W$ with respect to the grid $X_N$ in the sense of Proposition 3.5. Then

$$||W - w||_\mathcal{F} \leq ||W||_\mathcal{F}.$$
4 Error Estimates

In this section we prove three error estimates. More precisely, we estimate $|W'(x) - w'(x)|$ (Proposition 4.1) and $|W(x) - w(x)|$ (Proposition 4.2) in a neighborhood of $\overline{x}$. Moreover, we estimate $|D_n W(x) - D_n w(x)|$ for all points in the area where the grid points $X_N$ are set, cf. Theorem 4.4.

Proposition 4.1 Let $\Psi(x)$ be defined as in the Assumptions. Let $g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and let $W \in \mathcal{F}$.

For all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|W'(x) - w'(x)| \leq \varepsilon \quad \text{for } x \in \overline{B_\delta(\overline{x})} \quad (4.1)$$

holds for all approximations $w$ of $W$ with respect to any grid $X_N$ in the sense of Proposition 3.5.

Proof: There is a constant $c_1$ such that $\|Dg(x) - I\| \leq c_1$ holds for all $x \in \overline{B_1(\overline{x})}$. There is a constant $c_2$ such that $|\psi_1(\|g(x) - x\|)| \leq c_2$ holds for all $x \in \overline{B_1(\overline{x})}$, since $\psi_1$ and $g$ are continuous functions; for the definition of $\psi_1$ cf. Proposition 3.2. Set

$$\delta := \min \left(1, \frac{\varepsilon}{c_1 \|W\|_\mathcal{F} \sqrt{2c_2}} \right). \quad (4.2)$$

Denote the linear operator of the discrete orbital derivative by $Dw(x) := w(g(x)) - w(x)$. Let $x^* \in \overline{B_\delta(\overline{x})}$, and set $\lambda = \delta_{x^*} \circ D \in \mathcal{F}^*$ and $\mu = \delta_{\overline{x}} \circ D \in \mathcal{F}^*$, cf. Lemma 3.3. We have $\mu(W - w) = 0$, since $W'(\overline{x}) = 0 = w'(\overline{x})$ due to $g(\overline{x}) = \overline{x}$.

Hence,

$$|\lambda(W) - \lambda(w)| = |(\lambda - \mu)(W - w)| \leq \|\lambda - \mu\|_\mathcal{F}^* \cdot \|W - w\|_\mathcal{F}$$

by Proposition 3.6. Then we have with a similar calculation as in [3] – note that $g(\overline{x}) = \overline{x}$

$$\|\lambda - \mu\|_\mathcal{F}^2 = (\lambda - \mu)^x (\lambda - \mu)^y \Psi(x - y) = (\delta_{x^*} \circ D - \delta_{\overline{x}} \circ D)^x (\delta_{x^*} \circ D - \delta_{\overline{x}} \circ D)^y \Psi(x - y) = 2\psi(0) - 2\psi(\|g(x^*) - x^*\|).$$

Note that by Taylor’s Theorem there is a $\xi = \theta x^* + (1 - \theta)\overline{x}$ where $\theta \in [0, 1]$ such that

$$\psi(\|g(x^*) - x^*\|) = \psi(0) + \psi_1(\|g(\xi) - \xi\|) (g(\xi) - \xi)^T (Dg(\xi) - I)(x^* - \overline{x}).$$

Again by Taylor’s Theorem there is a $\xi' = \theta' \xi + (1 - \theta')\overline{x}$ where $\theta' \in [0, 1]$ such that

$$g(\xi) - \xi = g(\overline{x}) + Dg(\xi')(\xi - \overline{x}) - \xi$$

$$= (Dg(\xi') - I)(\xi - \overline{x}).$$
Altogether, we obtain
\[
\|\lambda - \mu\|_{\mathcal{F}^*}^2 \leq 2\psi_1(\|g(\xi) - \xi\|) \cdot \|g(\xi) - \xi\| \cdot \|Dg(\xi) - I\| \cdot \|x^* - \overline{x}\|
\]
\[
\leq 2c_2 \cdot \|Dg(\xi') - I\| \cdot \|Dg(\xi) - I\| \cdot \|x^* - \overline{x}\|^2
\]
\[
\leq 2c_1^2 c_2 \delta^2
\]
\[
\leq \frac{\epsilon^2}{\|W\|_{\mathcal{F}}^2}
\]
by (4.2) since \(\|x^* - \overline{x}\| \leq \delta\). Hence, we have \(\|\lambda - \mu\|_{\mathcal{F}^*} \leq \frac{\epsilon}{\|W\|_{\mathcal{F}}}\) and the proposition follows by (4.3). \(\Box\)

**Proposition 4.2** Let \(\Psi(x)\) be defined as in the Assumptions. Let \(g \in C^0(\mathbb{R}^d, \mathbb{R}^d)\) and let \(W \in \mathcal{F}\).

For all \(\epsilon > 0\) there is a \(\delta > 0\) such that
\[|W(x) - w(x)| \leq \epsilon \text{ for } x \in B_\delta(\overline{x})\] (4.4)
holds for all approximations \(w\) of \(W\) with respect to any grid \(X_N\) in the sense of Proposition 3.5.

**Proof:** Let \(m := \max_{r \in [0,1]} \left| \frac{d}{dr} \psi(r) \right|\) which exists by Proposition 3.2. Define
\[
\delta := \min \left(1, \frac{\epsilon^2}{2m\|W\|_{\mathcal{F}}^2} \right). \quad (4.5)
\]

Let \(x^* \in B_\delta(\overline{x})\), and set \(\lambda = \delta_{x^*} \in \mathcal{F}^*\) and \(\mu = \delta_{\overline{x}} \in \mathcal{F}^*\), cf. Lemma 3.3. We have \(\mu(W - w) = 0\), since \(W(\overline{x}) = 1 = w(\overline{x})\) by construction of \(w\), cf. (3.3). Hence,
\[
|\lambda(W) - \lambda(w)| = |(\lambda - \mu)(W - w)| \leq \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W - w\|_{\mathcal{F}}
\]
\[
\leq \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W\|_{\mathcal{F}} \quad (4.6)
\]
by Proposition 3.6. Denoting \(r := \|x^* - \overline{x}\| \leq \delta\) and using Taylor’s Theorem there is a \(\tilde{r} \in [0, r]\) such that
\[
\|\lambda - \mu\|_{\mathcal{F}^*}^2 = (\lambda - \mu)^2(\lambda - \mu)^y \Psi(x - y)
\]
\[
= (\delta_{x^*} - \delta_{\overline{x}})^2(\delta_{x^*} - \delta_{\overline{x}})^y \Psi(x - y)
\]
\[
= 2\psi(0) - 2\psi(r)
\]
\[
= -2\frac{d}{dr} \psi(\tilde{r})r
\]
\[
\leq 2m\delta \leq \frac{\epsilon^2}{\|W\|_{\mathcal{F}}^2}
\]
holds by (4.5). Hence, we have by (4.6) \(|\lambda(W) - \lambda(w)| \leq \epsilon\), which shows the proposition. \(\Box\)

For Theorem 4.4 we define the Fill distance of a grid.
Definition 4.3 (Fill distance) Let $K \subset \mathbb{R}^d$ be a compact set. Furthermore, let $X_N := \{x_1, \ldots, x_N\} \subset K$ be a grid (set of pairwise distinct points). The positive real number

$$h := h_{K,X_N} = \max_{y \in K} \min_{x \in X_N} \|x - y\|$$

is called the fill distance of $X_N$ in $K$. In particular, for all $y \in K$ there is a grid point $x_k \in X_N$ such that $\|y - x_k\| \leq h$.

Theorem 4.4 Let $\Psi(x)$ be defined as in the Assumptions. Let $g \in C^\sigma(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \geq 1$ and let $W \in \mathcal{F}$.

Let $K$ be a compact set. For all $H > 0$ there is a constant $C^* = C^*(K)$ such that: Let $X_N := \{x_1, \ldots, x_N\} \subset K$ be a grid with pairwise distinct points which are no fixed or periodic points and with fill distance $0 < h \leq H$ in $K$, and let $w \in C^2(\mathbb{R}^d,\mathbb{R})$ be the approximation of $W$ in the sense of Proposition 3.5. Then

$$|D_nW(x) - D_nw(x)| \leq C^*h \text{ holds for all } x \in K. \quad (4.7)$$

Proof: Let $x^* \in K$ and choose a grid point $\tilde{x} \in X_N$ such that $r := \|x^* - \tilde{x}\| \leq h$.

Set $\lambda = \delta_{x^*} \circ D_n \in \mathcal{F}^*$ and $\mu = \delta_{\tilde{x}} \circ D_n \in \mathcal{F}^*$, cf. Lemma 3.3. We have $\mu(W - w) = 0$ by definition of $w$, cf. (3.4). Hence,

$$|\lambda(W) - \lambda(w)| = \|\lambda - \mu\| \leq \|\lambda - \mu\|_F \cdot \|W - w\|_F \leq \|\lambda - \mu\|_F \cdot \|W\|_F$$

by Proposition 3.6. We obtain with a similar calculation as in Proposition 3.5

$$\|\lambda - \mu\|_F^2 = (\lambda - \mu)x^*(\lambda - \mu)y\Psi(x - y)$$

$$= (\delta_{x^*} \circ D_n - \delta_{\tilde{x}} \circ D_n)^x(\delta_{x^*} \circ D_n - \delta_{\tilde{x}} \circ D_n)^y\Psi(x - y)$$

$$= \psi(0)[n(\tilde{x})^2 + n(x^*)^2 + n(g(\tilde{x}))^2 + n(g(x^*))^2]$$

$$-2\psi(\|x^* - \tilde{x}\|)n(x^*)n(\tilde{x}) - 2\psi(\|x^* - g(\tilde{x})\|)n(g(x^*))n(g(\tilde{x}))$$

$$+2\psi(\|x^* - \tilde{x}\|)n(\tilde{x})n(x^*) + 2\psi(\|x^* - g(\tilde{x})\|)n(g(\tilde{x}))n(x^*)$$

$$+2\psi(\|x^* - \tilde{x}\|)n(\tilde{x})n(x^*) + 2\psi(\|x^* - g(\tilde{x})\|)n(g(\tilde{x}))n(x^*).$$

Due to the continuity of $g$, $Dg$, $n$, $\nabla n$ and Hess $n$, there are constants such that $\|g(x)\| \leq c_0$, $\|Dg(x)\| \leq c_1$, $n(x) \leq c_2$, $\|\nabla n(x)\| \leq c_3$ and $\|\text{Hess } n(x)\| \leq c_4$ hold for all $x \in \text{conv}(K \cup g(K))$.

Set $r := \|x^* - \tilde{x}\|$. Taylor’s Theorem implies the existence of $r_0 \in [0, r]$ such that

$$\psi(0) - \psi(\|x^* - \tilde{x}\|) = -\frac{d}{dr}\psi(r_0)r.$$
Moreover, there is a $\xi = \theta x^* + (1 - \theta)\bar{x}$ where $\theta \in [0, 1]$ such that $n(x^*) - n(\bar{x}) = \nabla n(\xi)(x^* - \bar{x})$. Hence,

$$
\begin{align*}
\psi(0)[n(\bar{x})^2 + n(x^*)^2] - 2\psi(\|x^* - \bar{x}\|)n(x^*)n(\bar{x}) \\
= \psi(\|x^* - \bar{x}\|)[n(x^*) - n(\bar{x})]^2 + [\psi(0) - \psi(\|x^* - \bar{x}\|)][n(x^*)^2 + n(\bar{x})^2] \\
\leq |\psi(r)|c_3^2r^2 + 2c_2^2\left|\frac{d}{dr}\psi(r_0)\right|r \\
= O(r^2)
\end{align*}
$$

(4.9)

for $r \to 0$ due to the properties of $\psi$, cf. 2. of Proposition 3.2.

In a similar way, Taylor’s Theorem implies the existence of $\xi_1 = \theta_1 x^* + (1 - \theta_1)\bar{x}$ where $\theta_1 \in [0, 1]$ such that

$$
\psi(0) - \psi(\|g(x^*) - g(\bar{x})\|) = -\psi_1(\|\xi_1\|)(g(\xi_1) - g(\bar{x}))^T Dg(\xi_1)(x^* - \bar{x}).
$$

Moreover, there are $\xi_2 = \theta_2 \xi_1 + (1 - \theta_2)\bar{x}$ and $\xi_3 = \theta_3 x^* + (1 - \theta_3)\bar{x}$ where $\theta_2, \theta_3 \in [0, 1]$ such that $g(\xi_1) - g(\bar{x}) = Dg(\xi_2)(\xi_1 - \bar{x})$ and $n(g(x^*)) - n(g(\bar{x})) = \nabla n(\xi_3)^T Dg(\xi_3)(x^* - \bar{x})$. Hence,

$$
\begin{align*}
\psi(0)[n(g(\bar{x}))^2 + n(g(x^*))^2] - 2\psi(\|g(x^*) - g(\bar{x})\|)n(g(x^*))n(\bar{x}) \\
= \psi(\|g(x^*) - g(\bar{x})\|)[n(g(x^*)) - n(\bar{x})]^2 \\
+ [\psi(0) - \psi(\|g(x^*) - g(\bar{x})\|)][n(g(x^*))^2 + n(\bar{x})^2] \\
\leq c_1^2c_3^2\psi(\|g(x^*) - g(\bar{x})\|)|r|^2 + 2c_1c_2^2|\psi_1(\|\xi_1\|)|r^2 \\
= O(r^2)
\end{align*}
$$

(4.10)

due to the properties of $\psi$, cf. 2. of Proposition 3.2.

For $h(x) = \psi(\|x - g(\bar{x})\|)n(x)$ we have $\nabla h(x) = \psi_1(\|x - g(\bar{x})\|)[n(x)(x - g(\bar{x})) + \psi(\|x - g(\bar{x})\|)\nabla n(x)]$. Hence, Taylor’s Theorem implies the existence of a $z_1 = \tau_1 x^* + (1 - \tau_1)\bar{x}$ where $\tau_1 \in [0, 1]$ such that

$$
\begin{align*}
\left[\psi(\|\bar{x} - g(\bar{x})\|)n(\bar{x}) - \psi(\|x^* - g(\bar{x})\|)n(x^*)\right]n(g(\bar{x})) \\
= \left[\psi_1(\|z_1 - g(\bar{x})\|)[n(z_1)(z_1 - g(\bar{x}))^T \\
+ \psi(\|z_1 - g(\bar{x})\|)\nabla n(z_1)^T](\bar{x} - x^*)\right]n(g(\bar{x}))
\end{align*}
$$

(4.11)

In a similar way we obtain

$$
\begin{align*}
\left[\psi(\|\bar{x} - g(x^*)\|)n(\bar{x}) - \psi(\|x^* - g(x^*)\|)n(x^*)\right]n(g(x^*)) \\
= \left[\psi_1(\|z_2 - g(x^*)\|)[n(z_2)(z_2 - g(x^*))^T \\
+ \psi(\|z_2 - g(x^*)\|)\nabla n(z_2)^T](\bar{x} - x^*)\right]n(g(x^*))
\end{align*}
$$

(4.12)

with $z_2 = \tau_2 x^* + (1 - \tau_2)\bar{x}$ where $\tau_2 \in [0, 1]$. 

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Subtracting (4.11) from (4.12) we obtain
\[
\psi(||g(\bar{x}) - x'||n(g(\bar{x}))n(x') - \psi(||g(\bar{x}) - \bar{x}||n(g(\bar{x})))n(\bar{x})
+\psi(||g(x') - \bar{x}||n(\bar{x})n(g(x')) - \psi(||g(x') - x'||n(x')n(g(x'))
= \n(z_2)n(g(x'))[\psi_1(||z_2 - g(x')||)(z_2 - g(x'))^T
- \psi_1(||z_1 - g(\bar{x})||)(z_1 - g(\bar{x}))^T](\bar{x} - x')
+ [n(z_2)n(g(x')) - n(z_1)n(g(\bar{x}))][\psi_1(||z_1 - g(\bar{x})||)(z_1 - g(\bar{x}))^T(\bar{x} - x')
+ [\psi(||z_2 - g(x')||)\nabla n(z_2)^T - \psi(||z_1 - g(\bar{x})||)\nabla n(z_1)^T](\bar{x} - x')n(g(x'))
+ \psi(||z_1 - g(\bar{x})||)\nabla n(z_1)^T(\bar{x} - x')n(g(\bar{x})).
\]

(4.13)

We will discuss each term of (4.13). For the first term we consider with \( z_3 = \tau_3(z_1 - g(\bar{x})) + (1 - \tau_3)(z_2 - g(x')) \) where \( \tau_3 \in [0, 1] \)
\[
\|\psi_1(||z_2 - g(x')||)(z_2 - g(x')) - \psi_1(||z_1 - g(\bar{x})||)(z_1 - g(\bar{x}))\|
\leq \|\psi_1(||z_2 - g(x')||)||z_2 - z_1 + (g(\bar{x}) - g(x'))||
+ |\psi_1(||z_1 - g(\bar{x})||) - \psi_1(||z_2 - g(x')||)| \cdot \|z_1 - g(\bar{x})||
= |\psi_1(||z_2 - g(x')||)(1 + c_1)r + |\psi_2(||z_3||)| \cdot \|z_3||(1 + c_1)r \cdot z_1 - g(\bar{x})||
\]

This term is \( O(r) \) for \( r \to 0 \) due to the properties of \( \psi \), cf. 2. of Proposition 3.2.

For the second term of (4.13) we consider
\[
|n(z_2)n(g(x')) - n(z_1)n(g(\bar{x}))| \leq n(z_2)|n(g(x')) - n(g(\bar{x}))| + n(g(\bar{x}))|n(z_2) - n(z_1)|
\leq c_2c_3(1 + c_1)r
= O(r) \text{ for } r \to 0.
\]

For the third term of (4.13) we have with \( z_4 = \tau_4(z_2 - g(x')) + (1 - \tau_4)(z_1 - g(\bar{x})) \) where \( \tau_4 \in [0, 1] \)
\[
\|\psi(||z_2 - g(x')||)\nabla n(z_2) - \psi(||z_1 - g(\bar{x})||)\nabla n(z_1)||
\leq |\psi(||z_2 - g(x')||)||\nabla n(z_2) - \nabla n(z_1)||
+ |\psi(||z_2 - g(x')||) - \psi(||z_1 - g(\bar{x})||)||\nabla n(z_1)||
\leq |\psi(||z_2 - g(x')||)|c_4r + |\psi_1(z_4)|| \cdot \|z_2 - z_1 + g(\bar{x}) - g(x')||c_3
\leq |\psi(||z_2 - g(x')||)|c_4r + |\psi_1(z_4)|| \cdot \|z_4||(1 + c_1)c_3r
= O(r)
\]

for \( r \to 0 \) due to the properties of \( \psi \), cf. 2. of Proposition 3.2.

For the last term of (4.13) we have
\[
|n(g(x')) - n(g(\bar{x}))| \leq c_2c_1r = O(r).
\]

Altogether, we obtain with (4.9), (4.10) and (4.13) \( \|\lambda - \mu\|^2_{x*} = O(r^2) \). Thus, (4.7) follows by (4.8). \( \square \)
5 Main Results

In order to show that the approximation \( v(x) = n(x)w(x) \) is a Lyapunov function, we show in Theorem 5.1 that \( v'(x) < 0 \) holds for all \( x \in K \setminus \{ \overline{\mathbf{x}} \} \). In Theorem 5.2 we show that each connected and bounded subset of the basin of attraction can be covered by a sublevel set of the function \( v \). Hence, we can determine each connected subset of the basin of attraction with a calculated Lyapunov function using the proposed method.

**Theorem 5.1** Let \( \mathbf{x} \) be a fixed point of \( x_{n+1} = g(x_n) \), where \( g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d) \) such that \( |\lambda| < 1 \) holds for all eigenvalues \( \lambda \) of \( Dg(\mathbf{x}) \).

We consider the radial basis function \( \Psi(x) = \psi_{l,k}(c||x||) \) with \( c > 0 \), where \( \psi_{l,k} \) denotes the Wendland function with \( k \in \mathbb{N} \) and \( l := \left[ \frac{d}{2} \right] + k + 1 \). Let \( \sigma \geq P \geq 2 + \sigma^* \), where \( \sigma^* := \frac{d+1}{2} + k \). Let \( K \) be a compact set such that \( \mathbf{x} \in \mathbb{N}^K \) and \( K \subset A(\mathbf{x}) \).

Let \( V \) be the Lyapunov function of Theorem 2.2 with \( V'(x) = -||x - \mathbf{x}||^2 \) and \( V(\mathbf{x}) = 0 \), and \( n(x) = \sum_{2 \leq |a| \leq P} c_a(x - \mathbf{x})^a + M||x - \mathbf{x}||^{2H} \) as in Definition 2.6, and let \( W(x) = \frac{V(g)}{n(x)} \in C^{P-2}(A(\mathbf{x}), \mathbb{R}) \) with \( W(\mathbf{x}) = 1 \), cf. Proposition 2.7.

Then there is a constant \( h' \), such that for all approximations \( w \in C^{2k}(\mathbb{R}^d, \mathbb{R}) \) of \( W \) with respect to a grid \( X_N \subset K \setminus \{ \mathbf{x} \} \) with fill distance \( h \leq h' \) in the sense of Proposition 3.5

\[
v'(x) < 0 \text{ holds for all } x \in K \setminus \{ \mathbf{x} \},
\]

where \( v(x) = w(x)n(x) \).

**Proof:** Let \( \overline{B} \) be an open set with \( K \subset \overline{B} \subset B \subset A(\mathbf{x}) \) and let \( \chi \in C_0^\infty(\mathbb{R}^d, [0, 1]) \) be a function with \( \chi(x) = 1 \) for \( x \in \overline{B} \) and \( \chi(x) = 0 \) for \( \mathbb{R}^d \setminus \overline{B} \). Thus, \( \chi \in C_0^\infty(\mathbb{R}^d) \subset \mathcal{F} \).

Set \( W_0 = W \cdot \chi; \) then \( W_0 \in C^P_{0}(\mathbb{R}^d, \mathbb{R}) \) and \( W_0(x) = W(x) \) holds for all \( x \in \overline{B} \).

Since \( P - 2 \geq \sigma^* = \frac{d+1}{2} + k \), we have \( W_0 \in H^{\frac{d+1}{2}+k}(\mathbb{R}^d) = \mathcal{F} \).

Set \( \epsilon = \frac{1}{8} \). We choose \( \delta > 0 \) so small that \( \overline{B}_\delta(\mathbf{x}) \subset K \) and

\[
\begin{align*}
n(g(x)) & \leq C||x - \mathbf{x}||^2 \quad (5.1) \\
n'(x) & \leq -\frac{1}{2}||x - \mathbf{x}||^2 \quad (5.2) \\
|W_0(x) - W_0(\mathbf{x})| & \leq \epsilon \quad (5.3) \\
|W_0'(x) - W_0'(\mathbf{x})| & \leq \epsilon \quad (5.4) \\
|W_0(x) - w(x)| & \leq \epsilon \quad (5.5) \\
|W_0'(x) - w'(x)| & \leq \epsilon \quad (5.6)
\end{align*}
\]

hold for all \( x \in \overline{B}_\delta(\mathbf{x}) \). This is possible due to Proposition 2.7 for \( g(\overline{B}_\delta(\mathbf{x})) \) – for (5.2) note that \( P > 2 \) and \( b'(x) = -||x - \mathbf{x}||^2 + o(||x - \mathbf{x}||^P) \) implies \( n'(x) = -||x - \mathbf{x}||^2 + o(||x - \mathbf{x}||^P) \) – the continuity of \( W_0 \) and \( W_0' \) at \( \mathbf{x} \), and by Propositions 4.1 and 4.2. Define \( h' := \min(1, \delta^2/C^*) \), where \( C^* \) is as in Theorem 4.4 for \( W_0, H = 1 \) and \( K \).
Theorem 5.2 Let $\bar{\alpha}$ be a fixed point of $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$ such that $|\lambda| < 1$ holds for all eigenvalues $\lambda$ of $Dg(\bar{\alpha})$.

We consider the radial basis function $\bar{\psi}(x) = \psi_{l,k}(c||x||)$ with $c > 0$, where $\psi_{l,k}$ denotes the Wendland function with $k \in \mathbb{N}$ and $l := \left\lfloor \frac{d}{2} \right\rfloor + k + 1$. Let $\sigma \geq P \geq 2 + \sigma^*$, where $\sigma^* := \frac{d+1}{2} + k$. Let $O_0$ be an open, bounded and connected set with $\bar{\alpha} \in O_0 \subset \overline{O_0} \subset A(\bar{\alpha})$.

Let $V$ be the Lyapunov function of Theorem 2.2 with $V'(x) = -||x - \bar{\alpha}||^2$ and $V(\bar{\alpha}) = 0$, and $n(x) = \sum_{2 \leq |\alpha| \leq P} c_\alpha (x - \bar{\alpha})^\alpha + M||x - \bar{\alpha}||^{2H}$ as in Definition 2.6, and let $W(x) = \frac{V(x)}{n(x)} \in C^{P-2}(\overline{A(\bar{\alpha})}, \mathbb{R})$ with $W(\bar{\alpha}) = 1$, cf. Proposition 2.7.

Then there is an open, bounded and connected set $\overline{\mathcal{B}} \subset A(\bar{\alpha})$ and an $h^* > 0$, such that for all approximations $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ of $W$ with respect to a grid $X_N \subset \overline{\mathcal{B}} \setminus \{\bar{\alpha}\}$ with fill distance $h < h^*$ in the sense of Proposition 3.5 there is an open, bounded and connected set $O$ with $O_0 \subset O$ such that for $v(x) = w(x)n(x)$ we have

\[ v'(x) < 0 \] for all $x \in K \setminus \{\bar{\alpha}\}$, and distinguish between the two cases $x \in \overline{B_5(\bar{\alpha})} \setminus \{\bar{\alpha}\}$ and $x \in K \setminus \overline{B_5(\bar{\alpha})}$.

Case 1: Let $x \in \overline{B_5(\bar{\alpha})} \setminus \{\bar{\alpha}\}$. Then

\[
v' (x) = \begin{align*}
n(g(x))w(g(x)) - n(x)w(x) \\
[w(g(x)) - w(x)]n(g(x)) + [n(g(x)) - n(x)]w(x) \\
= n(g(x))w'(x) + n'(x)w(x) \\
\leq C||x - \bar{\alpha}||^2 \left(||W'(\bar{\alpha})|| + \frac{\epsilon}{C} \right) - \frac{1}{2}||x - \bar{\alpha}||^2(W_0(x) - \epsilon) \\
\quad \text{by (5.1), (5.6), (5.2) and (5.5)} \\
\leq ||x - \bar{\alpha}||^2 \left(2\epsilon - \frac{1}{2}(1 - 2\epsilon) \right)
\end{align*}
\]

since $\epsilon = \frac{\delta}{6}$. This shows $v'(x) < 0$ for $x \in \overline{B_5(\bar{\alpha})} \setminus \{\bar{\alpha}\}$.

Case 2: Due to Theorem 4.4 and $h \leq h'$ we have $|V'(x) - v'(x)| = |D_nW_0(x) - D_nw(x)| \leq \delta^2$ for all $x \in K$. Then

\[
v'(x) \leq V'(x) + \delta^2 = -||x - \bar{\alpha}||^2 + \delta^2 < 0
\]

for all $x \in K \setminus \overline{B_5(\bar{\alpha})}$. This shows $v'(x) < 0$ and proves the theorem. \hfill \square
• $v(x) < R^*$ holds for all $x \in O$ and $v(x) = R^*$ holds for all $x \in \partial O$ for an $R^* \in \mathbb{R}^+$,

• $v'(x) < 0$ holds for all $x \in \overline{O} \setminus \{\bar{x}\}$.

**Proof:** Set $R := \max_{x \in \overline{O}} V(x) > 0$ and define the sets $K_1$ and $K_2$ as the closure of the connected components which include $\bar{x}$ of the following sets

$$
\{ x \in A(\bar{x}) \mid V(x) \leq R \} \quad \text{for } K_1
$$

$$
\{ x \in A(\bar{x}) \mid V(x) \leq R + 4 \} \quad \text{for } K_2,
$$

respectively. Denote by $B$ the connected component which includes $\bar{x}$ of $\{ x \in A(\bar{x}) \mid V(x) < R + 5 \}$. Then obviously $\overline{O} \subset K_1 \subset K_2 \subset B \subset \overline{B} \subset A(\bar{x})$ and $B$ is open; recall that the sets $K_1$, $K_2$ and $\overline{B}$ are compact, cf. Theorem 2.2. All these sets are positively invariant.

Let $\bar{B}$ be an open set with $\overline{B} \subset \bar{B} \subset A(\bar{x})$, e.g. $\bar{B} = \{ x \in A(\bar{x}) \mid V(x) < R + 6 \}$. Let $\chi \in C^0_c(\mathbb{R}^d, [0, 1])$ be a function with $\chi(x) = 1$ for $x \in \overline{B}$ and $\chi(x) = 0$ for $\mathbb{R}^d \setminus \bar{B}$. Thus, $\chi \in C^0_c(\mathbb{R}^d) \subset F$. Set $W_0 = W \cdot \chi$: then $W_0 \in C^{p-2}(\mathbb{R}^d, \mathbb{R})$ and $W_0(x) = W(x)$ holds for all $x \in \overline{B}$. Since $P - 2 \geq \sigma^*$, $W_0 \in H^{\frac{d+1}{2}+k}(\mathbb{R}^d) = F$.

For $\epsilon = \frac{1}{2}$ choose $0 < \delta \leq 1$ with Proposition 4.2 for $W_0$ such that $\overline{B}_\delta(\bar{x}) \subset O_0$ and such that for all $x \in \overline{B}_\delta(\bar{x})$ we have

$$
|W_0(x) - w(x)| \leq \frac{\epsilon}{C}, \quad (5.7)
$$

where $C$ was defined in 2. of Proposition 2.7 for $\overline{B}_1(\bar{x})$.

Choose $1 \geq r_0 > 0$ so small that

$$
U := \{ x \in A(\bar{x}) \mid V(x) < r_0 \} \subset \overline{B}_\delta(\bar{x}) \quad \text{holds.}
$$

We define a function $T: A(\bar{x}) \rightarrow \mathbb{N}_0$ denoting the minimal number $T(x) \in \mathbb{N}_0$ such that $S_T(x) \in U$. The function fulfills $T(x) = 0$ if and only if $x \in U$. Moreover, by definition $T'(x) \leq 0$ holds. Since $\overline{B} \subset A(\bar{x})$ is a compact set, there is a $\theta_0 > 0$ such that $S_{\theta_0} \overline{B} \subset U$. Thus, $0 \leq T(x) \leq \theta_0$ holds for all $x \in \overline{B}$.

Let $h' > 0$ be the constant from Theorem 5.1 for the set $\overline{B}$ and define $h^* := \min(h', (2\theta_0 C^*)^{-1})$ where $C^*$ was defined in Theorem 4.4 for the set $\overline{B}$ and $H = h'$. Thus,

$$
v'(x) < 0 \quad \text{holds for all } x \in \overline{B} \setminus \{\bar{x}\} \quad (5.8)
$$

and $|V'(x) - v'(x)| \leq \frac{1}{2\theta_0}$ holds for all $x \in \overline{B}$ \quad (5.9)

by Theorem 5.1 and Theorem 4.4.

Now let $x \in U \subset \overline{B}_\delta(\bar{x})$. Hence, $V(x) \in [0, r_0)$. For $v(x) = n(x) w(x)$ we have for all $x \in \overline{B}_\delta(\bar{x})$

$$
|V(x) - v(x)| = n(x) |W_0(x) - w(x)| \leq \max_{x \in \overline{B}_\delta(\bar{x})} n(x) |W_0(x) - w(x)| \leq \frac{1}{2}
$$

\[\leq C \delta^2\]

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by (5.7) since $\epsilon = \frac{1}{2}$ and $\delta \leq 1$. Thus, $v(x) \leq V(x) + \frac{1}{2} < r_0 + \frac{1}{2}$ and $v(x) \geq V(x) - \frac{1}{2} \geq -\frac{1}{2}$, i.e.

$$v(x) \in \left[-\frac{1}{2}, r_0 + \frac{1}{2}\right] \quad \text{for all } x \in U. \quad (5.10)$$

Now define $O$ as the connected component including $\overline{x}$ of

$$\{x \in B \mid v(x) < R + 2 =: R^*\}.$$ 

We will show that $K_1 \subset O \subset K_2$ holds. Then $O$ is bounded and $\partial O \subset K_2 \subset \overline{B}$, i.e. $v'(x) < 0$ holds for all $x \in \overline{O} \setminus \{\overline{x}\}$. Furthermore, $v(x) < R^*$ holds for all $x \in O$ and $v(x) = R^*$ holds for all $x \in \partial O \subset K_2$. On the other hand, this shows $O_0 \subset K_1 \subset O$, i.e. $O_0 \subset O$.

To prove $K_1 \subset O$ we let $x \in K_1$. Then we have in particular $\theta_0 \geq T(x) \geq 0$ and $V(y) \in [0, r_0)$ for all $y \in U$. Hence, we obtain

$$v(x) = v(S_T(x)x) - \sum_{k=0}^{T(x)-1} v'(S_kx)$$

$$< r_0 + \frac{1}{2} - \sum_{k=0}^{T(x)-1} \left(V'(S_kx) - \frac{1}{2\theta_0}\right) \quad \text{by (5.10) and (5.9)}$$

$$\leq V(S_T(x)x) - \sum_{k=0}^{T(x)-1} V'(S_kx) + r_0 + \frac{1}{2} + \frac{T(x)}{2\theta_0}$$

$$\leq V(x) + r_0 + 1 \leq R + 2 = R^*$$

since $r_0 \leq 1$. This shows $x \in O$.

For the inclusion $O \subset K_2$ we show that for $x \in \partial K_2$ we have $v(x) > R^*$. For $x \in \partial K_2 \subset \partial B$ we have $V(x) = R + 4$. With $T(x) \leq \theta_0$ and $V(y) \in [0, r_0)$ for all $y \in U$ we have

$$v(x) = v(S_T(x)x) - \sum_{k=0}^{T(x)-1} v'(S_kx)$$

$$\geq -\frac{1}{2} - \sum_{k=0}^{T(x)-1} \left(V'(S_kx) + \frac{1}{2\theta_0}\right) \quad \text{by (5.10) and (5.9)}$$

$$> V(S_T(x)x) - \sum_{k=0}^{T(x)-1} V'(S_kx) - r_0 - \frac{1}{2} - \frac{T(x)}{2\theta_0}$$

$$\geq V(x) - 1 - \frac{1}{2} - \frac{1}{2} = R + 4 - 2 = R^*,$$

i.e. $x \not\in O$. This proves the theorem. \qed
6 Examples

6.1 A Toy Example

Consider the difference equation

\[
\begin{align*}
    x_{n+1} &= \frac{1}{2} x_n + x_n^2 - y_n^2 \\
    y_{n+1} &= -\frac{1}{2} y_n + x_n^2
\end{align*}
\]

with fixed point \( \mathbf{x} = (0, 0) \). This is Example 1 of [3]. We have \( d = 2 \) and fix \( k = 1 \), thus \( \sigma^* = \frac{5}{2} \). A function \( n \) for \( P = 5 \) has been calculated in (2.6) and (2.7).

Now we approximate \( W(x) = \frac{V(x)}{n(x)} \), where \( V'(x) = -\|x\|^2 \), by \( w \). We use a hexagonal grid of the form \( 0.2 \left(i + \frac{j}{2}, j \sqrt{3} \right) \) with \( N = 118 \) points without the point \( \mathbf{x} \) \( (i = j = 0) \). Note that the grid used in Section 4.1 of [3] is of the form \( 0.2 \left(i - \frac{j}{2}, j \sqrt{3} \right) \) and consists of \( N = 119 \) points. The radial basis function is given by \( \Psi(x) = \psi_{3,1}(0.6\|x\|) \). For the function \( v(x) = w(x) n(x) \) we show the sign of \( v'(x) \) in Figure 1 as well as the sublevel set \( O_R = \{ x \in \mathbb{R}^2 \mid v(x) < R \} \) with \( R = 1.05 \); note that \( O \) is a subset of the basin of attraction \( A(0,0) \). This subset of the basin of attraction is of similar size as the one obtained in [3], Section 4.1. However, the discrete orbital derivative is now negative also in a neighborhood of \( \mathbf{x} \) in contrast to [3]. Figure 2 shows the local structure of \( v(x) \) and \( v'(x) \) near the equilibrium \( \mathbf{x} = (0, 0) \).

Figure 1: All figures correspond to Example (6.1). The black point \( \mathbf{x} = (0, 0) \) denotes the stable fixed point, whereas the grey points mark two unstable fixed points. Left: the grid with \( N = 118 \) points (black diamonds) and the sign of \( v'(x) \) (grey). Right: the sign of \( v'(x) \) (grey) and a sublevel set \( O_R \) (black) of \( v \) for \( R = 1.05 \), which is a subset of the basin of attraction \( A(0,0) \).
6.2 A model for the demand for education

As a second example we consider a model proposed in [9] to model the human decision for or against higher education and the development of wages. For the model it is assumed that in each time period \( n \) a continuum of agents of mass one is born that lives for two periods. They decide in the first time unit of their life, whether they invest into education. If they do so, they can work in the high-level sector, otherwise they have to work in the low-level sector for all their life.

If the agent \( i \) invests into education, the wages will be \( 1 - e_i \) in the first year and \( w^e_{n+1} \) in the following; note that the wages are set 1 in the low-level sector and \( w_n \) in the high-level sector at time \( n \). \( e_i \) reflects the costs of education for agent \( i \) and the costs are equally distributed over the continuum of agents. If the agent does not invest into education, his wages will be 1 in each year. The result is calculated by a utility function \( U(1 - e_i, w^e_{n+1}) \), \( U(1, 1) \), respectively. The utility function is given by the standard Cobb-Douglas utility function

\[
U(a, b) = a^\gamma b^{1-\gamma}
\]

with \( 0 < \gamma < 1 \) and we set \( \delta := \frac{1-\gamma}{\gamma} \).

The agent \( i \) thus will invest into education, if \( U(1 - e_i, w^e_{n+1}) > U(1, 1) \), where \( w^e_{n+1} \) is the expected wage in the following year; hence, all agents with \( e_i < e(w^e_{n+1}) \) will invest into education, where

\[
e(w^e_{n+1}) = 1 - (w^e_{n+1})^{-\delta}.
\]

We assume that there are two possibilities for the forecast of the expected wage.
of the following year $w_{n+1}^e$: the **adaptive agents** have the wage forecast

$$w_{n+1}^e = \rho w_n + (1 - \rho)w_{n-1}$$

with $0 \leq \rho \leq 1$, using a weighted average of the past two year’s wages. The **fixed point forecasters** (or steady state forecasters) use the fixed point $w^*$ of the dynamical system as their forecast; we assume that it costs the information cost $C_s > 0$ to find out this fixed point.

We denote the amount of agents following the fixed point forecast by $z_n$ and the adaptive agents by $1 - z_n$. The supply of high-skilled labor depending on the wage $w_{n+1}$ at time $n + 1$ is thus

$$l^s = (1 - z_n)e(\rho w_n + (1 - \rho)w_{n-1}) + z_n e(w^*) \quad (6.3)$$

The demand for high-skilled labour depending on the wage is given by

$$l^d(w) = \left(\frac{\alpha}{w}\right)^{1/(1 - \mu)} \quad (6.4)$$

with productivity parameter $\alpha > 0$ and $\mu = \frac{\delta - 1}{\delta}$. The market clearing condition $l^d(w_{n+1}) = l^s$ determines $w_{n+1}$ as a function of $w_n$, $w_{n-1}$ and $z_n$, i.e. cf. (6.3), (6.4) and (6.2)

$$w_{n+1} = \alpha \left[ (1 - z_n) [1 - (\rho w_n + (1 - \rho)w_{n-1})^{-\delta}] + z_n (1 - (w^*)^{-\delta}) \right]^{-1/\delta}$$

The amount of agents $z_n$, $1 - z_n$, respectively, following the different strategies will depend on how successful the respective strategies have been in the last year; this is called the generation overspill. Agents who choose the fixed-point strategy will regret their decision of education if they had earned more without education, i.e. if $e(w_{n+1}) \leq e \leq e(w^*)$. The regret is the difference between the possible utility $U(1, 1)$ and the actual one $U(1 - e, w_{n+1})$. Hence, considering the accumulated regret $R_f$ for the fixed point forecasters we obtain

$$R_f(w^*, w_{n+1}) = \int_{e(w_{n+1})}^{e(w^*)} R(e, w_{n+1}) \, de$$

where $R(e, w_{n+1}) = U(1, 1) - U(1 - e, w_{n+1})$.

Similarly we obtain the accumulated regret for the choice of the adaptive forecasters as

$$R_a(w_{n-1}, w_n, w_{n+1}) = \int_{e(w_{n+1})}^{e(\rho w_n + (1 - \rho)w_{n-1})} R(e, w_{n+1}) \, de.$$  

We define the difference $R^d = R_a - R_f = \int_{e(w^*)}^{e(\rho w_n + (1 - \rho)w_{n-1})} (1 - (1 - e)^{\gamma} w_{n+1}^{1 - \gamma}) \, de$. If $R^d$ is large, then the next generation prefers the fixed point strategy $f$, if it is small or negative, the next generation rather prefers the adaptive strategy $a$. The costs
Figure 3: We consider Example (6.5). The figure shows the stable fixed point \((x, y, z)\) (black point), the grid with \(N = 100\) points (black diamonds) and the level set \(v'(x, y, z) = 0\) (grey); note that the sign of \(v'\) is negative near the fixed point.

\(C_s\) of the fixed point strategy have a negative influence on the choice of the fixed point strategy. Altogether \(z_{n+1} = H(\beta(R^d - C_s))\), where \(H(x) = \frac{1}{1 + \exp(-x)}\). Note that the parameter \(\beta > 0\) measures how sensible the next generation is to the advice of the former one.

Altogether, we obtain the following model, where we set \(x_n = w_n^{-\delta}\) and \(y_n = x_{n-1}\), cf. [8], equation (10)

\[
\begin{align*}
  x_{n+1} &= G_s(x_n, y_n, z_n) \\
  y_{n+1} &= x_n \\
  z_{n+1} &= H(\beta(R^d(x_n, y_n, G_s(x_n, y_n, z_n)) - C_s))
\end{align*}
\]

with the functions, cf. equation (11) of [8]

\[
G_s(x, y, z) = \frac{1}{\alpha \delta}(1 - z) \left( 1 - (\rho x^{-1/\delta} + (1 - \rho) y^{-1/\delta})^{-\delta} \right) + \frac{1}{1 + \alpha \delta} z
\]

\[
R^d(x, y, \phi) = x^* - (\rho x^{-1/\delta} + (1 - \rho) y^{-1/\delta})^{-\delta}
\]

\[
\frac{1 + \delta}{2 + \delta} \phi^{-1/(1 + \delta)} \left( x^* \right)^{\frac{2+\delta}{2+\delta}} - (\rho x^{-1/\delta} + (1 - \rho) y^{-1/\delta})^{-\delta \frac{2 + \delta}{1 + \delta}}
\]

\[
H(x) = \frac{1}{1 + \exp(-x)}
\]

and \(x^* = (w^*)^{-\delta} = \frac{1}{1 + \alpha^*}\). By Proposition 7 of [8], \(\bar{x} = \bar{y} = x^*\) and \(\bar{z} = \frac{1}{1 + \exp(\beta C_s)}\) is a fixed point.

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Figure 4: All figures correspond to Example (6.5). The black point is the stable fixed point \((\bar{x}, \bar{y}, \bar{z})\). We show the level set \(v'(x, y, z) = 0\) (grey) together with the sublevel set \(O_R\) (brown) of \(v\) for \(R = 0.8\), which is a subset of the basin of attraction \(A(\bar{x}, \bar{y}, \bar{z})\). Left: the part with \(z \in [0, 0.4]\); note that this is the relevant part since the model is valid only for \(z \geq 0\). This part is also positively invariant, since \(z_{n+1} > 0\) by (6.5). Right: the part with \(z \in [-0.2, 0.1]\).

We choose the constants \(\alpha = \frac{3}{4}\), \(\delta = 2\), \(C_s = \frac{1}{4}\), \(\rho = 0.4\), and \(\beta = 10\). Since \(\alpha < 1\) and \(\rho < 1 - \alpha^\delta = \frac{7}{16}\), 3. of Proposition 7 implies, since \(\beta C_s < \ln\left(\frac{\alpha^\delta}{1 - \rho - \alpha^\delta}\right)\), that the fixed point is locally stable. However, there exists also the following stable period-three cycle, which lies outside the area of the model: \(x_1 = (0.160, 1.236, 0.473)\), \(x_2 = (0.845, 0.160, 0.101)\), \(x_3 = (1.236, 0.845, 0.194)\).

We are interested in the basin of attraction \(A(\bar{x}, \bar{y}, \bar{z})\). We have \(d = 3\) and fix \(k = 1\), thus \(\sigma^* = 3\). A function \(n\) for \(P = 5\) is given in Appendix A. We use a hexagonal grid of the form \((\bar{x}, \bar{y}, \bar{z}) + 0.08(i + \frac{1}{2} + \frac{k}{2}, j + \frac{1}{2}, k - \frac{1}{2})\) with \(99\) points without the point \(i = j = k = 0\) and with one additional point to obtain the Lyapunov function \(v\), so that \(N = 100\). We choose the radial basis function \(\Psi(x) = \psi_{3.1}(\frac{1}{2}||x||)\). The sign of \(v'(x)\) together with the grid is shown in Figure 3. In Figure 4 we show the level set \(v'(x) = 0\) (grey) as well as the sublevel set \(O_R = \{(x, y, z) \mid v(x, y, z) < R\}\) with \(R = 0.8\) (brown) which is a subset of the basin of attraction \(A(\bar{x}, \bar{y}, \bar{z})\). Note that the set \(O_R\) includes also points with negative \(z\)-values, whereas the model only makes sense for \(z \geq 0\). The set \(O_R \cap \{(x, y, z) \mid z \geq 0\}\) is shown in Figure 4, left, and is also a subset of \(A(\bar{x}, \bar{y}, \bar{z})\), since it is positively invariant; this can be seen from (6.5) since \(z_{n+1} = H(\zeta) = \frac{1}{1+\exp(-\zeta)} > 0\) for all \(\zeta \in \mathbb{R}\).
A  A function n for Section 6.2

A function n for the example of Section 6.2, cf. (6.5) is given by

\[ n(x, y, z) = \frac{(e^5 + 2e^{5/2} + 1)(31e^{5/2} + 15)}{-1525e^{15/2} + 21201e^5 + 24705e^{5/2} + 6075} (x - \overline{x})^2 \]

\[ + \frac{18432(1 + e^{5/2})e^5}{-1525e^{15/2} + 21201e^5 + 24705e^{5/2} + 6075} (x - \overline{x})(y - \overline{y}) \]

\[ + \frac{1135223e^{15/2} + 175125e^5 + 123525e^{5/2} + 30375}{5} (y - \overline{y})^2 \]

\[ - 1118.8301485(x - \overline{x})^3 - 318.2321415(x - \overline{x})(y - \overline{y}) \]

\[ - 37.36154017(x - \overline{x})^2(z - \overline{z}) + 25.35481402(x - \overline{x})(y - \overline{y})^2 \]

\[ - 167.3283992(x - \overline{x})(y - \overline{y})(z - \overline{z}) - 166.9291335(y - \overline{y})^2(z - \overline{z}) \]

\[ + 34.80534914(y - \overline{y})^3 \]

\[ - 467.59666833(x - \overline{x})^4 - 687.0022561(x - \overline{x})^3(y - \overline{y}) \]

\[ + 290.8657722(x - \overline{x})^3(z - \overline{z}) - 774.2211474(x - \overline{x})^2(y - \overline{y})^2 \]

\[ + 481.594964(x - \overline{x})^2(y - \overline{y})(z - \overline{z}) + 40.1403576(x - \overline{x})^2(z - \overline{z})^2 \]

\[ - 194.835032(x - \overline{x})(y - \overline{y})^3 - 182.5493215(x - \overline{x})(y - \overline{y})^2(z - \overline{z}) \]

\[ + 120.4210073(x - \overline{x})(y - \overline{y})(z - \overline{z})^2 - 536.3814964(y - \overline{y})^4 \]

\[ - 191.2350810(y - \overline{y})^3(z - \overline{z}) + 90.3157554(y - \overline{y})^2(z - \overline{z})^2 \]

\[ + 1138.088078(x - \overline{x})^5 + 1115.953760(x - \overline{x})^4(y - \overline{y}) \]

\[ + 6.907579760(x - \overline{x})^4(z - \overline{z}) - 1569.685708(x - \overline{x})^3(y - \overline{y})^2 \]

\[ + 1697.730028(x - \overline{x})^3(y - \overline{y})(z - \overline{z}) - 105.8818797(x - \overline{x})^3(z - \overline{z})^2 \]

\[ - 4208.607644(x - \overline{x})^2(y - \overline{y})^3 + 2603.716790(x - \overline{x})^2(y - \overline{y})^2(z - \overline{z}) \]

\[ - 9.294101910(x - \overline{x})^2(y - \overline{y})(z - \overline{z})^2 - 3963.856339(x - \overline{x})(y - \overline{y})^4 \]

\[ + 2286.60626(x - \overline{x})(y - \overline{y})^3(z - \overline{z}) + 418.695483(x - \overline{x})(y - \overline{y})^2(z - \overline{z})^2 \]

\[ - 190.3537367(y - \overline{y})^5 + 2287.846586(y - \overline{y})^4(z - \overline{z}) \]

\[ + 291.603610(y - \overline{y})^3(z - \overline{z})^2 \]

\[ + 5000((x - \overline{x})^2 + (y - \overline{y})^2 + (z - \overline{z})^2)^3. \]

B  Proof of Lemma 2.4

Proof: [of Lemma 2.4] By Remark 2.5, (2.2) can be solved by considering (2.3), which is a system of linear equations for \( c_n \) when considering each order of \( (x - \overline{x}) \). We will show that this system has a unique solution.

We will first transform the problem so that \( Dg(\overline{x}) = J \) is an upper diagonal matrix. Indeed, there is an invertible matrix \( S \) such that \( J = SDg(\overline{x})S^{-1} \) is the (complex) Jordan normal form of \( Dg(\overline{x}) \). In particular, \( J \) is an upper diagonal
matrix. Note that \( S \) and \( J \) are complex-valued matrices. However, if we can show that there is a unique solution \( \hat{h} \), it is obvious that all coefficients \( c_\alpha \) are in fact real. Define \( y := S(x - \overline{x}) \), then \( x = S^{-1} y + \overline{x} \). The iteration \( x_{n+1} = g(x_n) \) then is equivalent to \( y_{n+1} = S(x_{n+1} - \overline{x}) = S(g(x_n) - \overline{x}) = S(g(S^{-1} y_n + \overline{x}) - \overline{x}) =: f(y_n) \), and we have \( Df(y) = SDg(S^{-1} y + \overline{x}) S^{-1} \) and \( Df(0) = SDg(\overline{x}) S^{-1} = J \), which is an upper diagonal matrix. We will show that the following equation (B.1) has a unique solution \( \hat{h}(y) = \sum_{2 \leq |\alpha| \leq P} c_\alpha y^\alpha \).

\[
\begin{align*}
\hat{h}(f(y)) - \hat{h}(y) &= -y^T Cy + o(\|y\|^P),
\end{align*}
\] (B.1)

where \( C = (S^{-1})^T S^{-1} \).

Note that \( \hat{h}(y) = \sum_{2 \leq |\alpha| \leq P} c_\alpha y^\alpha \) is a solution of (B.1) if and only if \( \hat{h}(x) := \hat{h}(S(x - \overline{x})) = \sum_{2 \leq |\alpha| \leq P} c_\alpha (S(x - \overline{x}))^\alpha = \sum_{2 \leq |\alpha| \leq P} c_\alpha' (x - \overline{x})^\alpha \) is a solution of (2.2), since we have with \( y = S(x - \overline{x}) \)

\[
\begin{align*}
\hat{h}(f(y_n)) - \hat{h}(y_n) &= \hat{h}(y_{n+1}) - \hat{h}(y_n) \\
&= \hat{h}(S(x_{n+1} - \overline{x})) - \hat{h}(S(x_n - \overline{x})) \\
&= \hat{h}(x_{n+1}) - \hat{h}(x_n) \\
\text{and } - y^T Cy + o(\|y\|^P) &= -\|S^{-1} y\|^2 + o(\|S^{-1} y\|^P) \\
&= -\|x - \overline{x}\|^2 + o(\|x - \overline{x}\|^P).
\end{align*}
\]

Hence, we have to show that there exists a unique solution \( \hat{h}(y) = \sum_{2 \leq |\alpha| \leq P} c_\alpha y^\alpha \) of (B.1), i.e. cf. Remark 2.5

\[
\sum_{2 \leq |\alpha| \leq P} c_\alpha \left[ \left( J y + \sum_{2 \leq |\beta| \leq P-1} \frac{\partial^\beta f(0)}{\beta!} y^\beta \right)^\alpha - y^\alpha \right] = -y^T Cy + o(\|y\|^P). \tag{B.2}
\]

Note that \( \partial^\beta f(0) = (\partial^\beta f_1(0), \ldots, \partial^\beta f_d(0))^T \) is a vector, \( C \) is a symmetric matrix and \( P \geq 2 \). We consider the terms order by order in \( y \). The lowest appearing order is two and the terms of this order in \( y \) of both sides of (B.2) are

\[
\sum_{|\alpha|=2} c_\alpha [(Jy)^\alpha - y^\alpha] = -y^T Cy. \tag{B.3}
\]

Writing the terms of order two as \( \sum_{|\alpha|=2} c_\alpha y^\alpha = y^T By \), (B.3) becomes \( J^T BJ - B = -C \). Since the eigenvalues \( \lambda \) of \( J \) are the eigenvalues of \( Dg(\overline{x}) \) and thus satisfy \( |\lambda| < 1 \), this equation has a unique solution \( B \); the proof for the complex-valued matrix \( J \) is the same as for the real case, cf. [10] or Lemma 2.5 of [3].

Now we show by induction with respect to \( k := |\alpha| \leq P \) that the constants \( c_\alpha \) are uniquely determined by (B.2): For \( |\alpha| = 2 \) this has just been done. Now let \( P \geq |\alpha| = k \geq 3 \). Consider (B.2): the terms of order \( \leq k - 1 \) satisfy the equation by induction. Now consider the terms of order \( |\alpha| = k \): all terms of order \( k \) are contained in the following expression:

\[
\sum_{2 \leq |\alpha| \leq k-1} c_\alpha \left( J y + \sum_{2 \leq |\beta| \leq P-1} \frac{\partial^\beta f(0)}{\beta!} y^\beta \right)^\alpha + \sum_{|\alpha|=k} c_\alpha [(Jy)^\alpha - y^\alpha]. \tag{B.4}
\]
The constants $c_\alpha$ with $2 \leq |\alpha| \leq k - 1$ are fixed. We will show that there is a unique solution for the constants $c_\alpha$ with $|\alpha| = k$ such that

$$\sum_{2 \leq |\alpha| \leq k - 1} c_\alpha \left( J y + \sum_{2 \leq |\beta| \leq k - 1} \frac{\partial^{P-1} f(0)}{\partial \beta^{P-1}} y^\beta \right)^\alpha + \sum_{|\alpha| = k} c_\alpha [(J y)^\alpha - y^\alpha]$$

has no terms of order $k$. Since all $c_\alpha$ in the left-hand term of (B.4) are known and all $c_\alpha$ in the right-hand term are unknown, this is equivalent to an inhomogeneous system of linear equations. It has a unique solution, if and only if the corresponding homogeneous system has only the zero solution. Therefore, we study the corresponding homogeneous problem

$$\sum_{|\alpha| = k} c_\alpha [(J y)^\alpha - y^\alpha] = 0,$$  \hspace{1cm} \text{(B.5)}

and show that $c_\alpha = 0$ is its only solution.

$J$ is an upper diagonal matrix such that all eigenvalues $\lambda$ satisfy $|\lambda| < 1$ since $J$ is the Jordan Normal Form of $Dg(\bar{x})$. The eigenvalues are on the diagonal and thus $|J_{ii}| < 1$ holds for all $1 \leq i \leq d$. Hence,

$$J y = \left( J_{11} y_1 + J_{12} y_2 + \ldots + J_{1d} y_d, J_{22} y_2 + \ldots + J_{2d} y_d, \ldots, J_{dd} y_d \right)^T.$$  \hspace{1cm} \text{(B.6)}

We prove by induction that all coefficients $c_\alpha$ in (B.5) vanish. Note that $|\alpha| = k$. We introduce an order on $\bar{A} := \{ \alpha \in \mathbb{N}_0^d \mid \alpha_i \in \{0, \ldots, k\} \text{ for all } i \in \{1, \ldots, d\} \}$. Note that $\{ \alpha \in \mathbb{N}_0^d \mid |\alpha| = k \} \subset \bar{A}$. The order $||\alpha||$ on $\bar{A}$ is such that $\alpha$ is the $(k + 1)$-adic expansion of $||\alpha||$, i.e.

$$\mathbb{N}_0 \ni ||\alpha|| = \sum_{l=1}^d \alpha_l (k + 1)^{d-l}.$$  \hspace{1cm} \text{(B.5)}

Now we start the induction with respect to $||\alpha||$. The minimal $\alpha$ with $|\alpha| = k$ is $\alpha = (k, 0, \ldots, 0)$. The coefficient of $y_1^k$ in (B.5) is, due to (B.6), $c_{(k, 0, \ldots, 0)} (J_{11}^k - 1)$. Since $|J_{11}^k| = |J_{11}|^k < 1$, we have $c_{(k, 0, \ldots, 0)} = 0$.

Now we assume that all coefficients $c_\alpha$ with $|\alpha| = k$ and $||\alpha|| \leq A^*$ for some $A^* \in \mathbb{N}$ are zero. Let $\beta \in \mathbb{N}_0^d$ be minimal with $||\beta|| > A^*$ and $|\beta| = k$. We will show that $c_\beta = 0$. Consider the coefficient of $y^\beta$ in (B.5). Due to (B.6), only the terms $\sum_{|\alpha| = k} c_\alpha (J y)^\alpha$ with

$$\beta_1 \leq \alpha_1 \text{ (to obtain } y_1^\beta_1\text{)}$$

$$\beta_1 + \beta_2 \leq \alpha_1 + \alpha_2 \text{ (to obtain } y_1^\beta_2\text{)}$$

$$\vdots$$

$$\sum_{l=1}^{d-1} \beta_l \leq \sum_{l=1}^{d-1} \alpha_l \text{ (to obtain } y_l^\beta_{d-1}\text{)}$$

$$\sum_{l=1}^{d} \beta_l \leq \sum_{l=1}^{d} \alpha_l \text{ (to obtain } y_d^\beta_d\text{)}$$

contribute to terms with $y^\beta$. 27
We will show that \( \|\alpha\| \leq \|\beta\| \). The last inequality is \( k = k \), since \( |\alpha| = |\beta| = k \).

By subtracting the last inequality from the one before, we obtain \( \alpha_d \leq \beta_d \). If \( \alpha_d < \beta_d \), then \( \|\alpha\| < \|\beta\| \). If \( \alpha_d = \beta_d \), then \( \sum_{i=1}^{d-1} \beta_i = \sum_{i=1}^{d-1} \alpha_i \) and we can proceed by subtracting the second-last from the third-last inequality, and obtain \( \alpha_{d-1} \leq \beta_{d-1} \) etc.

Since \( c_\alpha = 0 \) holds for all \( \|\alpha\| < \|\beta\| \) by induction, the only term left with \( y^\beta \) is the one with \( \|\alpha\| = \|\beta\| \), i.e.

\[
c_\beta[z^\beta y^\beta - y^\beta] = 0
\]

where \( z = (J_{11}, J_{22}, \ldots, J_{dd})^T \). Since \( |J_{ii}| < 1 \) for all \( i = 1, \ldots, d \), we have \( |z^\beta| < \max_{i=1,\ldots,d} |J_{ii}|^{k} < 1 \) and thus \( z^\beta - 1 \neq 0 \). Hence, we can conclude \( c_\beta = 0 \).

We show that \( h \) is the Taylor polynomial of \( V \) of order \( P \): Let \( V(x) = T(x) + \tilde{\varphi}(x) \) with \( \tilde{\varphi}(x) = o(\|x - \overline{x}\|^P) \), where \( T(x) = \sum_{0 \leq |\alpha| \leq P} d_\alpha (x - \overline{x})^\alpha \) is the Taylor polynomial of \( V \) of order \( P \). Then

\[
\begin{align*}
-\|x - \overline{x}\|^2 &= V'(x) \\
&= V(g(x)) - V(x) \\
&= T(g(x)) - T(x) + \tilde{\varphi}(g(x)) - \tilde{\varphi}(x) \\
&= T'(x) + o(\|x - \overline{x}\|^P) \tag{B.7}
\end{align*}
\]

since \( \tilde{\varphi}(x) = o(\|x - \overline{x}\|^P) \) by definition and \( \tilde{\varphi}(g(x)) = o(\|x - \overline{x}\|^P) \) as we show now:

Set \( m := \max_{y \in B_{\delta}(\overline{x})} \|Dg(y)\| \). Let \( \epsilon > 0 \). Then there is a \( 0 < \delta < 1 \) such that

\[
\left| \frac{\tilde{\varphi}(x)}{\|x - \overline{x}\|^P} \right| < \frac{\epsilon}{m} \tag{B.8}
\]

for all \( x \in B_{\delta}(\overline{x}) \setminus \{\overline{x}\} \), since \( \tilde{\varphi}(x) = o(\|x - \overline{x}\|^P) \). Since \( g(x) - g(\overline{x}) = g(x) - \overline{x} = o(1) \) by Taylor’s Theorem, for the above \( \delta > 0 \), there is a \( 0 < \delta' < 1 \) such that \( \|g(x) - \overline{x}\| < \delta \), i.e. \( g(x) \in B_{\delta}(\overline{x}) \) for all \( x \in B_{\delta'}(\overline{x}) \). On the other hand, the mean value theorem yields \( \|g(x) - g(\overline{x})\| \leq m \|x - \overline{x}\| \), i.e.

\[
\frac{1}{\|x - \overline{x}\|} \leq m \|g(x) - g(\overline{x})\| \tag{B.9}
\]

for all \( x \in B_{\delta'}(\overline{x}) \setminus \{\overline{x}\} \). With (B.8) and (B.9) we conclude that for all \( \epsilon > 0 \) there is a \( 0 < \delta' < 1 \) such that

\[
\left| \frac{\tilde{\varphi}(g(x))}{\|x - \overline{x}\|^P} \right| \leq m \left| \frac{\tilde{\varphi}(g(x))}{\|g(x) - \overline{x}\|^P} \right| < \epsilon
\]

for all \( x \in B_{\delta'}(\overline{x}) \setminus \{\overline{x}\} \). This shows \( \tilde{\varphi}(g(x)) = o(\|x - \overline{x}\|^P) \).

Note that \( V(\overline{x}) = 0 \), and hence \( d_0 = 0 \). The terms of order one are also zero.

Indeed, denote \( d_i := c_i \), for \( i = 1, \ldots, d \) such that \( \sum_{|\alpha|=1} d_\alpha (x - \overline{x}) = d^T (x - \overline{x}) \). Then (B.7) implies \( d^T Dg(\overline{x})(x - \overline{x}) - d^T (x - \overline{x}) = 0 \), i.e. \( d^T (Dg(\overline{x}) - I) = 0 \).

Assume in contradiction that \( d \neq 0 \). Then \( \det(Dg(\overline{x}) - I) = 0 \), i.e. that \( \lambda = 1 \) is an eigenvalue of \( Dg(\overline{x}) \), which is a contradiction. By the uniqueness of \( h \), we have \( \sum_{0 \leq |\alpha| \leq P} d_\alpha (x - \overline{x})^\alpha = h(x) \), since it satisfies (2.2). \( \square \)
References


